Exam in Numerical Analysis FMN050, 2017–08–24

Exam duration 14:00–18:00. In order to pass, a minimum of 15 points is required on the exam. The (preliminary) grade requirements are
grade 3 ≥ 15 p, **grade 4** ≥ 21 p and **grade 5** ≥ 26 p.

You are allowed to use a pocket calculator, but no other material of any kind. Please answer the problems in Swedish or English.

Problems

1. Let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \text{where} \quad A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}.$$

Compute the condition number of the matrix A with respect to the 1-norm. How large can the solution's relative error $\|\delta x\|_1/\|x\|_1$ be when solving $Ax = b$ with the data error $\|\delta b\|_1/\|b\|_1 = 0.2$? **(2p)**

2. Consider the linear system $Ax = b$, where the matrix A can be factorized as $PA = LU$ with

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 5 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solve $Ax = b$, with $b = (1, 1, 3)^{\mathrm{T}}$, by first pivoting and thereafter performing one forward and one backward substitution. **(3p)**

3. We would like to approximate the intersections of the circle $x^2 + y^2 = 2$ and the curve $y = x^3$, by employing Newton's method.

a) Formulate the problem as a system of nonlinear equations of the form $f(x, y) = 0$. How many solutions are there? **(1p)**

b) Construct the Jacobian matrix $f'(x, y)$. **(1p)**

c) Introduce a suitable notation and write out Newton's method for the problem at hand. **(2p)**

4. Construct the Newton interpolation polynomial that interpolates the data below. **(3p)**

x	0	1	2
y	1	-1	2

5. We would like to fit a straight line $y(x) = c_0 + c_1 x$ to the data below.

x	0	1	2	4
y	1	0	-1	-2

a) Derive the overdetermined system $Ac \approx b$ which the coefficients $c = (c_0, c_1)^{\mathrm{T}}$ need to fulfill. **(1p)**

b) Approximate a solution to the overdetermined system by the least squares method. **(2p)**

6. Consider the integral

$$\int_0^1 x^2(x+1)\,\mathrm{d}x.$$

a) Approximate the integral by employing Simpson's rule and dividing the interval $[0, 1]$ into $N = 2$ equally sized intervals. **(2p)**

b) Why is the approximation error equal to zero when applying Simpson's rule to the integral above? **(1p)**

7. We want to approximate the solution to the two-point boundary value problem

$$y''(x) + y(x)\big(y(x) + 1\big) = 0,$$

with boundary values $y(0) = 0$ and $y(2) = 2$. Introduce a suitable notation and discretize the problem by a standard second order method, i.e., $y''(x)$ should be discretized as in Project 2. Construct the system of *nonlinear* equations that have to be solved, and make sure to include the boundary conditions. All details, such as the number of equations, the grid, the step size Δx, etc., must be clearly stated. **(4p)**

8. Consider the initial value problem

$$y'(t) = -1000\, y(t), \quad t > 0,$$

with the initial value $y(0) = 2$.

a) Write down the implicit Euler discretization of the initial value problem. **(1p)**

b) Derive for which step sizes Δt your discretization is stable. **(2p)**

c) Explain what is meant by the statement "the implicit Euler method has a convergence order of $p = 1$". **(1p)**

d) State an implicit discretization of the initial value problem that has a convergence order of $p = 2$. **(1p)**

9. Consider the MATLAB code below.

```
A = [1 3 ; 2 2]
y = [-1 ; 1]
n = 12
for k = 1:n
     yh = y/norm(y,2)
     y = A*yh
     sigma = yh'*y
end
sigma_out = sigma
y_out = y/norm(y,2)
```

Which numerical method is implemented in the code? What does `sigma_out` and `y_out` approximate, respectively? **(3p)**

LYCKA TILL!