Convergence analysis of Schwarz-like methods for degenerate elliptic-parabolic equations

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Abstract Convergence is proven for Schwarz-like methods applied to degenerate elliptic-parabolic equations with a p-structure. This family of PDEs, e.g., arises when modelling nonlinear diffusion processes. The Schwarz-like approximation methods are based on decomposing the space-time domain into overlapping subdomains, which enables parallel implementations. The methods are derived by introducing a pseudo-time component and applying time integrators of splitting type, which are time stepped towards infinity. This approach of decomposing the space-time domain is related to Schwarz waveform relaxation methods, but the methods considered here have the advantage that they can be proven to converge when applied to nonlinear parabolic, or even degenerate elliptic-parabolic, PDEs. We prove convergence by deriving a nonlinear framework based on the abstract theory for monotone operators and the existence theory for degenerate elliptic-parabolic equations.

Keywords Convergence analysis · degenerate elliptic-parabolic equations · Schwarz methods · overlapping domain decompositions · splitting time integrators

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1 Introduction

Consider the degenerate elliptic-parabolic problem

$$\begin{cases} \partial_t(\gamma u) - \nabla \cdot \alpha(t, \nabla u) + \beta(t,u) + f(t) = 0 & \text{in } \Omega \times (0,T), \\ \alpha(t,\nabla u)\cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \gamma u(0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \in \mathbb{R}^d$ is a bounded Lipschitz domain with the boundary $\partial\Omega$, and $\mathbf{n}$ denotes the unit outward normal vector of $\partial\Omega$. Here, γ is a nonnegative function and $\alpha(t,\nabla u)$ may vanish for nonzero values of ∇u. That is, for subsets of Ω, where $\gamma(x) = 0$ the equation switches from parabolic to elliptic. Furthermore, in contrast to linear parabolic problems, the degeneracy feature that α may be zero for nonzero arguments describes a diffusion process with finite speed of propagation. A survey of applications involving degenerate elliptic-parabolic equations can be found in [15, Chapters 8 and 12]. We consider a homogeneous initial condition for the sake of simplicity. For the non-homogeneous case, see [16, Chapter 3] for details.

As these equations are both nonlinear and require implicit time discretizations, they are numerically challenging and require large-scale computations. A common approach to facilitate such computations is to employ domain decomposition methods, which enable parallel implementations. In the context of elliptic equations, a domain decomposition method consists of first decomposing the equation's domain into subdomains. Thereafter, the method solves the elliptic equation on each subdomain and communicates the results via the overlaps, or boundaries, of the adjacent subdomains. The classical application of these methods to parabolic equations is to first apply a time integrator and thereafter apply domain decomposition to the elliptic problems arising at each time step. For a general introduction, we refer to [14,20]. Domain decomposition methods can also be directly incorporated into time integrators by interpreting the decomposition procedure as a splitting scheme, see, e.g., [4, 11,21].

Combining domain decomposition with time integration allows for parallelization in space, but inherently prevents parallel implementations in time. To remedy this, one can consider Schwarz waveform relaxation (SWR) methods, where the full space-time domain is decomposed. These space-time decomposition methods have been proposed in the contexts of parallel time integrators; surveyed in [9], space-time finite elements; surveyed in [17], and parabolic problems with a spatial domain given by a union of domains with very different material properties [2]. There are several studies concerning the convergence and other theoretical aspects of space-time decomposition methods applied to non-degenerate parabolic equations, see, e.g., [6,7, 8,10].

However, results for SWR methods are lacking from the literature when considering more general degenerate parabolic equations. This absence of numerical results is likely explained as follows. The standard convergence analysis for SWR methods applied to linear equations [14] relies on interpreting the method as a projection procedure, which is not applicable for nonlinear equations. For degenerate elliptic equations, i.e., (1) with $\gamma = 0$, one can derive convergence analyses by relying on the coercivity of the degenerate elliptic operator [5,18]. This property is lost for degenerate parabolic equations, as the operator $u \mapsto \partial_t(\gamma u)$ is just monotone and not necessarily coercive. Hence, if one is to introduce approximation schemes similar to the SWR methods for degenerate parabolic equations, or even degenerate elliptic-parabolic equations, then a more general framework is needed.

The goal of our study is therefore to introduce a new family of domain decomposition schemes that have similar properties to the SWR methods and prove their convergence when applied to degenerate elliptic-parabolic equations. To achieve this, we will derive the new schemes in Section 2 and cast them into an abstract Cauchy framework based on monotone operators in Section 4. The convergence can then be proven in the abstract elliptic framework of [13,19], as done in Section 5. We derive in Section 6 how degenerate elliptic-parabolic equations with a p-structure can be incorporated into our abstract Cauchy framework. The analysis presented here is restricted to the continuous case, and the design of a parallel space-time finite element method based on the new Schwarz-like methods will be considered elsewhere.

2 Constructing Schwarz-like schemes

We will derive a family of approximation schemes that are Schwarz-like in the sense that they decompose the space-time domain $\Omega \times (0,T)$ into overlapping subdomains. To this end, consider a collection of subsets $\{\Omega_\ell\}_{\ell=1}^q$ that satisfies $\cup_{\ell=1}^q \Omega_\ell = \Omega$. Each subset Ω_ℓ is either a Lipschitz domain, or a union of pairwise disjoint Lipschitz domains $\{\Omega_{\ell,j}\}_{j=1}^r$ such that $\cup_{j=1}^r \Omega_{\ell,j} = \Omega_\ell$. Over the subdomain $\{\Omega_\ell\}_{\ell=1}^q$ we introduce a partition of unity $\{\chi_\ell\}_{\ell=1}^q$, where the weights satisfy

$$\chi_\ell(x) > 0 \text{ a.e. } x \in \Omega_\ell, \quad \chi_\ell(x) = 0 \text{ a.e. } x \in \Omega \setminus \Omega_\ell, \quad \text{and} \quad \sum_{\ell=1}^q \chi_\ell = 1.$$

Given four partition of unities $(\chi_\ell^\alpha, \chi_\ell^\beta, \chi_\ell^\gamma, \chi_\ell^f)$ over $\{\Omega_\ell\}_{\ell=1}^q$, we can formally decompose our degenerate elliptic-parabolic equation (1) as

$$\begin{aligned} \partial_t(\gamma u) - \nabla \cdot \alpha(t, \nabla u) + \beta(t,u) + f(t) &= \sum_{\ell=1}^q \partial_t(\chi_\ell^\gamma \gamma u) - \nabla \cdot \chi_\ell^\alpha \alpha(t,\nabla u) + \chi_\ell^\beta \beta(t,u) + \chi_\ell^f f(t) \\ &= \sum_{\ell=1}^q \mathscr{F}_\ell u = 0. \end{aligned} \tag{2}$$

Here, each term $\mathscr{F}_\ell u$ is a function with support in the space-time domain $\Omega_\ell \times (0,T)$. With the formulation (2), we can derive approximation schemes in the same way as done in [19] for elliptic equations. That is, we introduce a pseudo-time $\xi \in [0,\infty)$ and the corresponding evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} v + \sum_{\ell=1}^q \mathscr{F}_\ell v = 0, \quad v(0) \text{ given.} \tag{3}$$

If the maps $\mathscr{F}_\ell$ fulfill some form of monotonicity property, we obtain that $v(\xi) \to u$ as $\xi \to \infty$. Hence, we can approximate u by applying a standard time integrator of splitting type to (3) and then time step towards infinity.

For the special case $q = 2$, we can employ the Peaceman–Rachford time integrator in the above approach. This gives an approximation scheme of the form: find $\{u_1^n, u_2^n\}_{n\in\mathbb{N}}$ such that

$$\begin{cases} (sI + \mathscr{F}_1)u_1^{n+1} = (sI - \mathscr{F}_2)u_2^n, \\ (sI + \mathscr{F}_2)u_2^{n+1} = (sI - \mathscr{F}_1)u_1^{n+1}, \end{cases} \tag{4}$$

where u_2^0 is an initial guess and u_ℓ^n is an approximation of u in $\Omega_\ell \times (0,T)$. Here, the method parameter $s > 0$ can be interpreted as $1/\tau$, with τ denoting the pseudo-time step. If we apply the same procedure to the Douglas–Rachford method, then the corresponding approximation scheme reads

$$\begin{cases} (sI + \mathscr{F}_1)u_1^{n+1} = (sI - \mathscr{F}_2)u_2^n, \\ (sI + \mathscr{F}_2)u_2^{n+1} = su_1^{n+1} + \mathscr{F}_2 u_2^n. \end{cases} \tag{5}$$

These methods are suitable in this context as they preserve the equilibrium of the pseudo-time problem. In the general case $q \geq 2$, we can use the additive splitting method to obtain the approximation scheme

$$\begin{cases} (sI + \mathscr{F}_\ell)u_\ell^{n+1} = su^n \quad \text{for } \ell = 1,\dots,q, \\ u^{n+1} = \dfrac{1}{q}\displaystyle\sum_{\ell=1}^{q} u_\ell^{n+1}, \end{cases} \tag{6}$$

where u^n is an approximation of u in $\Omega \times (0,T)$. While this method does not necessarily preserve the equilibrium of the equation, it is straightforward to parallelize the additive splitting method.

One can of course create many other approximation schemes by choosing other splitting integrators. However, we will focus on (4) to (6), as they already have quite different convergence properties and require different tailored convergence proofs. Note that it is of little use to consider higher-order splitting methods, as degenerate elliptic-parabolic equations typically have solutions of low regularity.

3 Preliminaries

In the following analysis, for a real Banach space V, we denote its dual space by V^*. When inserting an element $v \in V$ into $u \in V^*$, we use the notation for the dual pairing $u(v) = \langle u, v\rangle_{V^*\times V}$. In the case that there exists a Hilbert space H such that V is densely embedded into H then we have a Gelfand triplet setting $V \hookrightarrow H \hookrightarrow V^*$ and $\langle u, v\rangle_{V^*\times V} = (u,v)_H$ for $u \in H, v \in V$.

Let $G\colon V \to V^*$ be a possibly nonlinear operator. We call G *bounded* if it maps bounded sets in V into bounded sets in V^*. If there exists a function $k\colon V \to [0,\infty)$ such that

$$\langle Gu - Gv, u - v\rangle_{V^*\times V} \geq k(u - v) \quad \text{for all } u, v \in V,$$

then the operator G is referred to as *k-monotone*. If $k \equiv 0$, then G is just *monotone*. The operator G is *hemicontinuous* if $\varepsilon \mapsto \langle G(u+\varepsilon v), w\rangle_{V^*\times V}$ is continuous for $\varepsilon \in [0,1]$ and all $u,v,w \in V$. The operator G is *coercive* if $\langle Gu,u\rangle_{V^*\times V} \to \infty$ as $\|u\|_V \to \infty$. We call G *symmetric* if $\langle Gu,v\rangle_{V^*\times V} = \langle Gv,u\rangle_{V^*\times V}$ for all $u,v \in V$.

Additionally, for some statements, we consider G as a possibly unbounded operator on a Hilbert space H. In this setting, an operator $G\colon D(G) \subseteq H \to H$, is called *accretive* if

$$(Gu - Gv, u - v)_H \geq 0 \quad \text{for all } u,v \in D(G).$$

Moreover, it is called *maximal* if $R\big(sI + G\big) = \mathscr{H}$ for all $s > 0$.

Throughout the paper, c and C will denote generic positive constants.

4 Abstract Cauchy framework

Let H be a Hilbert space and V be a separable, reflexive Banach space such that V is densely embedded into H. For a given $p \in [2,\infty)$ consider the induced spaces

$$\mathscr{H} = L^2(0,T;H) \quad \text{and} \quad \mathscr{V} = L^p(0,T;V).$$

By [12, Chapter II.2], we then have the identifications $\mathscr{V}^* \cong L^{p/(p-1)}(0,T;V^*)$ and

$$\langle u,v\rangle_{\mathscr{V}^*\times\mathscr{V}} = \int_0^T \langle u(t),v(t)\rangle_{V^*\times V}\,\mathrm{d}t,$$

as well as the Gelfand triplet $\mathscr{V} \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{V}^*$. Furthermore, consider a family of operators $\{A(t)\}_{t\in(0,T)}$, where $A(t)\colon V \to V^*$ is not necessarily linear, and the single bounded linear operator $M\colon H \to H$. The induced operators $\mathscr{A}\colon \mathscr{V} \to \mathscr{V}^*$ and $\mathscr{M}\colon \mathscr{H} \to \mathscr{H}$ are then given by

$$(\mathscr{A}u)(t) = A(t)u(t) \quad \text{and} \quad (\mathscr{M}u)(t) = Mu(t) \quad \text{for a.e. } t \in (0,T),$$

respectively. The main tool to analyze degenerate elliptic-parabolic equations, is to observe the following. The translation semigroup $\{S(\tau)\}$ on $\mathscr{V}^*$ defined as

$$S(\tau)u(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \tau, \\ u(t-\tau) & \text{for } \tau < t \leq T, \end{cases} \tag{7}$$

is generated by $-\mathrm{d}/\mathrm{d}t\colon D(\mathrm{d}/\mathrm{d}t) \subseteq \mathscr{V}^* \to \mathscr{V}^*$, where

$$\begin{aligned} D(\mathrm{d}/\mathrm{d}t) &= \{u \in \mathscr{V}^* : \lim_{\tau\to 0^+} \frac{1}{\tau}\big(I - S(\tau)\big)u = \mathrm{d}/\mathrm{d}t u \in \mathscr{V}^*\} \\ &= \{u \in W^{1,p/(p-1)}(0,T;V^*) : u(0) = 0\}. \end{aligned}$$

Compare with [16, Proposition 5.1]. Note that $W^{1,p/(p-1)}(0,T;V^*) \hookrightarrow C([0,T];V^*)$, by [15, Lemma 7.1], i.e., the pointwise evaluation in $D(\mathrm{d}/\mathrm{d}t)$ is well defined. We also define the space

$$\mathscr{W} = \{u \in \mathscr{V} : \mathscr{M}u \in D(\mathrm{d}/\mathrm{d}t)\}.$$

For a given $f \in \mathscr{V}^*$, one can now consider the nonlinear Cauchy problem of finding $u \in \mathscr{W}$ such that

$$\mathscr{F}u = (\mathrm{d}/\mathrm{d}t\mathscr{M} + \mathscr{A})u + f = 0 \quad \text{in } \mathscr{V}^*, \tag{8}$$

or equivalently, finding a solution to

$$-\int_0^T \big(Mu(t), \partial_t v(t)\big)_H \,\mathrm{d}t + \langle \mathscr{A}u + f, v\rangle_{\mathscr{V}^*\times\mathscr{V}} = 0 \tag{9}$$

for all $v \in W^{1,p}(0,T;V)$ with $v(T) = 0$. See [16, Chapter 3] for details on the equivalent formulations of Cauchy problems.

Definition 1 Consider the spaces V, H, which induce $\mathscr{H}, \mathscr{V}, \mathscr{W}$, together with the operators $M\colon H \to H$, $\mathscr{A}\colon \mathscr{V} \to \mathscr{V}^*$. If

1. V is a separable, reflexive Banach space that is densely embedded in the Hilbert space H;
2. M is linear, bounded, monotone, and symmetric;
3. $\mathscr{A}$ is bounded, k-monotone, hemicontinuous, and coercive,

then the problem set $(V, H, M, \mathscr{A})$ is *proper*.

Lemma 1 *If $(V, H, M, \mathscr{A})$ is proper, then*

$$\langle \mathrm{d}/\mathrm{d}t\mathscr{M}u, u\rangle_{\mathscr{V}^*\times\mathscr{V}} \geq 0 \quad \textit{for all } u \in \mathscr{W}.$$

Proof Let $u \in \mathscr{W}$ and observe that

$$\langle \mathrm{d}/\mathrm{d}t\mathscr{M}u, u\rangle_{\mathscr{V}^*\times\mathscr{V}} = \lim_{\tau\to 0^+} \frac{1}{\tau}\langle \big(I - S(\tau)\big)\mathscr{M}u, u\rangle_{\mathscr{V}^*\times\mathscr{V}}. \tag{10}$$

Furthermore, as M is monotone and symmetric, the bilinear form $\langle M\cdot, \cdot\rangle$ satisfies the Cauchy-Schwarz inequality

$$|\langle Mu, v\rangle_{V^*\times V}| \leq \langle Mu, u\rangle_{V^*\times V}^{1/2} \langle Mv, v\rangle_{V^*\times V}^{1/2} \qquad \text{for all } u, v \in V.$$

This observation implies that for every $u \in \mathscr{W}$, we have

$$\begin{aligned}
\langle S(\tau)\mathscr{M}u,u\rangle_{\mathscr{V}^*\times\mathscr{V}} &= \int_\tau^T \langle Mu(t-\tau), u(t)\rangle_{V^*\times V}\,\mathrm{d}t \\
&\leq \int_\tau^T \langle Mu(t-\tau), u(t-\tau)\rangle_{V^*\times V}^{1/2} \langle Mu(t), u(t)\rangle_{V^*\times V}^{1/2}\,\mathrm{d}t \\
&\leq \Big(\int_0^{T-\tau} \langle Mu(t), u(t)\rangle_{V^*\times V}\,\mathrm{d}t\Big)^{1/2} \Big(\int_\tau^T \langle Mu(t), u(t)\rangle_{V^*\times V}\,\mathrm{d}t\Big)^{1/2} \\
&\leq \langle \mathscr{M}u, u\rangle_{\mathscr{V}^*\times\mathscr{V}}.
\end{aligned} \tag{11}$$

Combining this with (10) gives that sought after bound. □

As $-\mathrm{d}/\mathrm{d}t$ generates a contraction semigroup on $\mathscr{V}^*$ satisfying (11), compare [16, Chapter III.5], one has for every proper set $(V,H,M,\mathscr{A})$, every $f \in \mathscr{V}^*$ in (8), and every $g \in \mathscr{V}^*$ that the equation

$$\mathscr{F}u = g \quad \text{in } \mathscr{V}^* \tag{12}$$

has a unique solution $u \in \mathscr{W}$. This is one of the main results from the existence theory for degenerate elliptic-parabolic equations, which is due to [3]. For an English version see [16, Proposition III.6.2]. Next, consider the domain

$$D(\mathscr{F}) = \{u \in \mathscr{W} : \mathscr{F}u \in \mathscr{H}\}.$$

together with the restricted operator $\mathscr{F}\colon D(\mathscr{F}) \subseteq \mathscr{H} \to \mathscr{H}$, which may be unbounded in $\mathscr{H}$. For any $s > 0$, the problem set $(V,H,M,sI+\mathscr{A})$ is also proper. The observation $\mathscr{H} \subset \mathscr{V}^*$ together with the existence result (12) then implies that the restriction of $\mathscr{F}$ is maximal. Furthermore, by Lemma 1 and the k-monotonicity of $\mathscr{A}$, the restriction of $\mathscr{F}$ is accretive. As a direct consequence, the resolvent

$$(sI+\mathscr{F})^{-1}\colon \mathscr{H} \to D(\mathscr{F}) \subseteq \mathscr{H},$$

is a well-defined, nonexpansive operator for every $s > 0$. In order to decompose (8), we assume the following.

Assumption 1 *The sets $(V,H,M,\mathscr{A})$ and $(V_\ell,H_\ell,M_\ell,\mathscr{A}_\ell)$, for $\ell = 1,\dots,q$, and the linear, bounded operators $E_\ell\colon H_\ell \to H$, $R_\ell\colon H \to H_\ell$ fulfill*

1. *all problem sets are proper;*
2. *$R_\ell E_\ell = I$ on H_ℓ and $(E_\ell u,v)_H = (u,R_\ell v)_{H_\ell}$ for all $u \in H_\ell, v \in H$;*
3. *$Mu = \sum_{\ell=1}^q E_\ell M_\ell R_\ell u$ for all $u \in H$.*

The induced operators $\mathscr{E}_\ell\colon \mathscr{H}_\ell \to \mathscr{H}$ and $\mathscr{R}_\ell\colon \mathscr{H} \to \mathscr{H}_\ell$, given by

$$(\mathscr{E}_\ell u)(t) = E_\ell u(t) \quad \text{and} \quad (\mathscr{R}_\ell u)(t) = R_\ell u(t) \quad \text{for a.e. } t \in (0,T),$$

respectively, are then linear and bounded. The second and third statements of Assumption 1 also hold for the pairs $\mathscr{E}_\ell,\mathscr{R}_\ell$ in combination with $\mathscr{M}$ and $\mathscr{M}_\ell$.

For given functionals $f_\ell \in \mathscr{V}_\ell^*$, $\ell = 1,\dots,q$, we can define the domains

$$D(\mathscr{F}_\ell) = \{u \in \mathscr{H} : \mathscr{R}_\ell u \in \mathscr{W}_\ell \text{ and } (\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)\mathscr{R}_\ell u + f_\ell \in \mathscr{H}_\ell\}$$

together with the operators $\mathscr{F}_\ell\colon D(\mathscr{F}_\ell) \subseteq \mathscr{H} \to \mathscr{H}$, given by

$$\mathscr{F}_\ell u = \mathscr{E}_\ell\big((\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)\mathscr{R}_\ell u + f_\ell\big) \quad \text{for } u \in D(\mathscr{F}_\ell).$$

Lemma 2 *If Assumption 1 holds then the operator $\mathscr{F}_\ell$ is maximal accretive on $\mathscr{H}$.*

Proof By assumption, the set $(\mathscr{V}_\ell, \mathscr{H}_\ell, \mathscr{M}_\ell, sI + \mathscr{A}_\ell)$ is proper and the operator

$$u_\ell \mapsto (\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)u_\ell + f_\ell \tag{13}$$

is maximal accretive on $\mathscr{H}_\ell$ with domain $\{u_\ell \in \mathscr{W}_\ell : (\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)u_\ell + f_\ell \in \mathscr{H}_\ell\}$.

Hence, for each $g \in \mathscr{H}$ there exists a unique u_ℓ in the domain such that

$$(sI + \mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)u_\ell + f_\ell = \mathscr{R}_\ell g.$$

The function $u = \mathscr{E}_\ell u_\ell + s^{-1}(I - \mathscr{E}_\ell\mathscr{R}_\ell)g$ is then in $D(\mathscr{F}_\ell)$, as $\mathscr{R}_\ell u = u_\ell$, and

$$\begin{aligned}(sI + \mathscr{F}_\ell)u &= s\mathscr{E}_\ell u_\ell + (I - \mathscr{E}_\ell\mathscr{R}_\ell)g \\ &\quad + \mathscr{E}_\ell\Big((\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)\big(\mathscr{R}_\ell\mathscr{E}_\ell u_\ell + s^{-1}\mathscr{R}_\ell(I - \mathscr{E}_\ell\mathscr{R}_\ell)g\big) + f_\ell\Big) \\ &= \mathscr{E}_\ell\big((sI + \mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)u_\ell + f_\ell\big) + (I - \mathscr{E}_\ell\mathscr{R}_\ell)g = g.\end{aligned}$$

That is, $\mathscr{F}_\ell$ is maximal.

Let $u, v \in D(\mathscr{F}_\ell)$ and set $\mathscr{R}_\ell u = u_\ell, \mathscr{R}_\ell v = v_\ell$. As the operator (13) is accretive,

$$\begin{aligned}(\mathscr{F}_\ell u - \mathscr{F}_\ell v, u - v)_{\mathscr{H}} &= \big(\mathscr{E}_\ell(\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)\mathscr{R}_\ell u - \mathscr{E}_\ell(\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)\mathscr{R}_\ell v, u - v\big)_{\mathscr{H}} \\ &= \big((\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)u_\ell - (\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell + \mathscr{A}_\ell)v_\ell, u_\ell - v_\ell\big)_{\mathscr{H}_\ell} \geq 0.\end{aligned}$$

Hence, $\mathscr{F}_\ell$ is accretive. □

Assumption 2 *The spaces V, V_ℓ and the operators R_ℓ, $\ell = 1, \dots, q$, fulfill*

1. *R_ℓ is a bounded operator from V to V_ℓ;*
2. *if $u \in H$ such that $R_\ell u \in V_\ell$ for all $\ell = 1, \dots, q$, then $u \in V$ and the bound $\|u\|_V^p \leq C\sum_{\ell=1}^q \|R_\ell u\|_{V_\ell}^p$ holds.*

It directly follows that Assumption 2 also holds for the induced spaces $\mathscr{V}, \mathscr{V}_\ell$ and operators $\mathscr{R}_\ell$.

Corollary 1 *Let Assumptions 1 and 2 be valid. If $u \in \mathscr{H}$ and $\mathscr{R}_\ell u \in \mathscr{W}_\ell$ for all $\ell = 1, \dots, q$, then $u \in \mathscr{W}$ and*

$$\langle \mathrm{d}/\mathrm{d}t\mathscr{M}u, v\rangle_{\mathscr{V}^* \times \mathscr{V}} = \sum_{\ell=1}^q \langle \mathrm{d}/\mathrm{d}t\mathscr{M}_\ell\mathscr{R}_\ell u, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell}$$

for all $v \in \mathscr{V}$.

Proof For a u fulfilling the hypothesis we have, by Assumption 2, that $u \in \mathscr{V}$ and it remains to show that $\mathscr{M}u \in D(\mathrm{d}/\mathrm{d}t)$. To this end, we observe that $\mathscr{R}_\ell u \in \mathscr{W}_\ell$ and the boundedness of $\mathscr{R}_\ell \colon \mathscr{V} \to \mathscr{V}_\ell$ implies

$$\langle z, \cdot\rangle_{\mathscr{V}^* \times \mathscr{V}} = \sum_{\ell=1}^q \langle \mathrm{d}/\mathrm{d}t\mathscr{M}_\ell\mathscr{R}_\ell u, \mathscr{R}_\ell(\cdot)\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} \in \mathscr{V}^*.$$

The definition (7) of $\{S(\tau)\}$ gives that

$$\begin{aligned}\langle \tfrac{1}{\tau}(I-S(\tau))\mathscr{M}u,v\rangle_{\mathscr{V}^*\times\mathscr{V}} &= \big(\tfrac{1}{\tau}(I-S(\tau))\mathscr{M}u,v\big)_{\mathscr{H}}\\ &= \sum_{\ell=1}^{q}\frac{1}{\tau}\int_0^T \big(E_\ell M_\ell R_\ell u(t),v(t)\big)_H\,\mathrm{d}t - \frac{1}{\tau}\int_\tau^T \big(E_\ell M_\ell R_\ell u(t-\tau),v(t)\big)_H\,\mathrm{d}t\\ &= \sum_{\ell=1}^{q}\big(\tfrac{1}{\tau}(I-S(\tau))\mathscr{M}_\ell\mathscr{R}_\ell u,\mathscr{R}_\ell v\big)_{\mathscr{H}_\ell} = \sum_{\ell=1}^{q}\langle \tfrac{1}{\tau}(I-S(\tau))\mathscr{M}_\ell\mathscr{R}_\ell u,\mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^*\times\mathscr{V}_\ell}\end{aligned}$$

for all $u\in\mathscr{H}, v\in\mathscr{V}$. Consider $z_\ell(\tau)=1/\tau(I-S(\tau))\mathscr{M}_\ell\mathscr{R}_\ell u$ and $z_\ell = \mathrm{d}/\mathrm{d}t\mathscr{M}_\ell\mathscr{R}_\ell u$. As $\mathscr{R}_\ell u\in\mathscr{W}_\ell$, we have $z_\ell(\tau)\to z_\ell$ in $\mathscr{V}_\ell^*$as $\tau\to 0^+$. Hence, it follows that

$$\|\tfrac{1}{\tau}(I-S(\tau))\mathscr{M}u - z\|_{\mathscr{V}^*}\le \sup_{v\in\mathscr{V}\setminus\{0\}}\frac{1}{\|v\|_{\mathscr{V}}}\sum_{\ell=1}^{q}|\langle z_\ell(\tau)-z_\ell,\mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^*\times\mathscr{V}_\ell}|\to 0$$

as $\tau\to 0^+$. That is, $\mathscr{M}u\in D(\mathrm{d}/\mathrm{d}t)$ and $\mathrm{d}/\mathrm{d}t\mathscr{M}u = z$. □

Note that Corollary 1 implies the inclusion $\cap_{\ell=1}^{q}D(\mathscr{F}_\ell)\subseteq\mathscr{W}$.

Assumption 3 *The identity*

$$\begin{aligned}\mathscr{F}u =&\mathrm{d}/\mathrm{d}t\mathscr{M}u+\mathscr{A}u+f\\ =&\sum_{\ell=1}^{q}\mathscr{E}_\ell(\mathrm{d}/\mathrm{d}t\mathscr{M}_\ell\mathscr{R}_\ell u+\mathscr{A}_\ell\mathscr{R}_\ell u+f_\ell) = \sum_{\ell=1}^{q}\mathscr{F}_\ell u\end{aligned}$$

in $\mathscr{H}$ holds for every $u\in\cap_{\ell=1}^{q}D(\mathscr{F}_\ell)$.

From Assumption 3 it is clear that $\cap_{\ell=1}^{q}D(\mathscr{F}_\ell)\subseteq D(\mathscr{F})$, but the assumption does not imply equality in general. In order to proceed with the analysis, we therefore assume the following additional regularity property.

Assumption 4 *The solution to $\mathscr{F}u=0$ satisfies $u\in\cap_{\ell=1}^{q}D(\mathscr{F}_\ell)$.*

5 Abstract method convergence

We will combine the abstract Cauchy framework in Section 4 together with the elliptic convergence results derived in [13, Proposition 1] and [19, Theorem 3.1]. As these results are central to our analysis, we give the proofs in the current notation.

Theorem 1 *Consider the Peaceman–Rachford (4) or Douglas–Rachford (5) approximation $\{u_1^n,u_2^n\}_{n\in\mathbb{N}}$ of the solution u to the nonlinear Cauchy problem $\mathscr{F}u=0$. If Assumptions 1 to 4 hold and $u_2^0\in D(\mathscr{F}_2)$ then*

$$\lim_{n\to\infty} k_1(u_1^n-u)+k_2(u_2^n-u)=0$$

for every method parameter $s>0$.

Proof We begin with the Peaceman–Rachford case. The regularity $u \in D(\mathscr{F}_1) \cap D(\mathscr{F}_2)$ implies that $\mathscr{F}_1 u = -\mathscr{F}_2 u$. If $u_2^n \in D(\mathscr{F}_2)$ then

$$\begin{aligned} u_1^{n+1} &= (sI + \mathscr{F}_1)^{-1}(sI - \mathscr{F}_2)u_2^n \in D(\mathscr{F}_1) \quad \text{and} \\ u_2^{n+1} &= (sI + \mathscr{F}_2)^{-1}(sI - \mathscr{F}_1)u_1^{n+1} \in D(\mathscr{F}_2). \end{aligned}$$

As $u_2^0 \in D(\mathscr{F}_2)$, we have by induction that $\{u_1^n, u_2^n\}_{n\in\mathbb{N}} \subset D(\mathscr{F}_1) \times D(\mathscr{F}_2)$. Let

$$v^n = (sI + \mathscr{F}_2)u_2^n, \quad v = (sI + \mathscr{F}_2)u, \quad w^n = (sI - \mathscr{F}_2)u_2^n \quad \text{and} \quad w = (sI - \mathscr{F}_2)u.$$

This notation implies the relations

$$\begin{gathered} u = \frac{v+w}{2s}, \quad u_2^n = \frac{v^n + w^n}{2s}, \quad u_1^{n+1} = \frac{v^{n+1} + w^n}{2s}, \\ \mathscr{F}_2 u = \frac{v-w}{2}, \quad \mathscr{F}_2 u_2^n = \frac{v^n - w^n}{2}, \quad \mathscr{F}_1 u = \frac{w-v}{2}, \quad \mathscr{F}_1 u_1^{n+1} = \frac{w^n - v^{n+1}}{2}. \end{gathered}$$

The accretivity of $\mathscr{F}_\ell$ then gives the two bounds

$$\begin{aligned} 0 \leq (\mathscr{F}_2 u_2^n - \mathscr{F}_2 u, u_2^n - u)_{\mathscr{H}} &= \tfrac{1}{4s}\big((v^n - v) - (w^n - w), (v^n - v) + (w^n - w)\big)_{\mathscr{H}} \\ &= \tfrac{1}{4s}\big(\|v^n - v\|_{\mathscr{H}}^2 - \|w^n - w\|_{\mathscr{H}}^2\big) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq (\mathscr{F}_1 u_1^{n+1} - \mathscr{F}_1 u, u_1^{n+1} - u)_{\mathscr{H}} \\ &= \tfrac{1}{4s}\big((w^n - w) - (v^{n+1} - v), (w^n - w) + (v^{n+1} - v)\big)_{\mathscr{H}} \\ &= \tfrac{1}{4s}\big(\|w^n - w\|_{\mathscr{H}}^2 - \|v^{n+1} - v\|_{\mathscr{H}}^2\big). \end{aligned}$$

The two bounds then yield $\|v^{n+1} - v\|_{\mathscr{H}}^2 \leq \|w^n - w\|_{\mathscr{H}}^2 \leq \|v^n - v\|_{\mathscr{H}}^2$. This shows that the real valued sequence $\{\|v^n - v\|_{\mathscr{H}}^2\}_{n\in\mathbb{N}}$ is monotonously decreasing and bounded from below by zero. Thus, it converges and it follows that

$$\|v^n - v\|_{\mathscr{H}}^2 - \|v^{n+1} - v\|_{\mathscr{H}}^2 \to 0 \quad \text{as } n \to \infty.$$

This combined with the two bounds above and (11), implies that

$$0 \leq \sum_{\ell=1}^{2} k_\ell(u_\ell^n - u) \leq \sum_{\ell=1}^{2} (\mathscr{F}_\ell u_\ell^n - \mathscr{F}_\ell u, u_\ell^n - u)_{\mathscr{H}} \to 0$$

as $n \to \infty$. The convergence proof for the Douglas–Rachford scheme follows in the same fashion and is therefore omitted. □

For the additive splitting (6), the parameter s needs to be chosen more carefully, as a function of N. An optimal choice of s is parameter dependent, compare [19, Theorem 5.1] for more details. For the sake of simplicity, we choose $s = C\sqrt{N}$ that fulfills [19, Remark 3.1] and guarantees convergence while keeping the notation compact.

Theorem 2 *Consider the additive splitting (6) approximation* $\{u^n\}_{n=1}^N$ *of the solution* u *to the nonlinear Cauchy problem* $\mathscr{F}u = 0$*. If Assumptions 1 to 4 hold and* $k_\ell \geq c\|\cdot\|^2_{\mathscr{H}}$ *then it follows that*

$$\lim_{N\to\infty} \|u^N - u\|_{\mathscr{H}} = 0$$

for the parameter choice $s = C\sqrt{N}$*.*

Proof Let $v_\ell^{n+1} = u_\ell^{n+1} - u$ and $v^n = u^n - u$. Then (6) gives

$$s(v_\ell^{n+1} - v^n) + (\mathscr{F}_\ell u_\ell^{n+1} - \mathscr{F}_\ell u) = -\mathscr{F}_\ell u \in \mathscr{H}.$$

By applying $\langle \cdot, s v_\ell^{n+1}\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell}$ to the equation above, we obtain

$$s^2(v_\ell^{n+1} - v^n, v_\ell^{n+1})_{\mathscr{H}} + s\langle \mathscr{F}_\ell u_\ell^{n+1} - \mathscr{F}_\ell u, v_\ell^{n+1}\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} = (-\mathscr{F}_\ell u, s v_\ell^{n+1})_{\mathscr{H}}.$$

The first term to the right can be rewritten as

$$\begin{aligned} s^2(v_\ell^{n+1} - v^n, v_\ell^{n+1})_{\mathscr{H}} &= \frac{s^2}{2}(v_\ell^{n+1} - v^n, v_\ell^{n+1})_{\mathscr{H}} + \frac{s^2}{2}(v_\ell^{n+1} - v^n, v_\ell^{n+1} - v^n)_{\mathscr{H}} \\ &\quad + \frac{s^2}{2}(v_\ell^{n+1} - v^n, v^n)_{\mathscr{H}} \\ &= \frac{s^2}{2}\|v_\ell^{n+1}\|^2_{\mathscr{H}} + \frac{s^2}{2}\|v_\ell^{n+1} - v^n\|^2_{\mathscr{H}} - \frac{s^2}{2}\|v^n\|^2_{\mathscr{H}}, \end{aligned}$$

and the bound $k_\ell \geq c\|\cdot\|^2_{\mathscr{H}}$ then yields the inequality

$$s(s+2c)\|v_\ell^{n+1}\|^2_{\mathscr{H}} - s^2\|v^n\|^2_{\mathscr{H}} + s^2\|v_\ell^{n+1} - v^n\|^2_{\mathscr{H}} \leq 2(-\mathscr{F}_\ell u, s v_\ell^{n+1})_{\mathscr{H}}.$$

Taking the average of these inequalities, for $\ell = 1, \ldots, q$, and noting that

$$\|v^{n+1}\|^2_{\mathscr{H}} \leq \frac{1}{q}\sum_{\ell=1}^{q} \|v_\ell^{n+1}\|^2_{\mathscr{H}},$$

by the convexity of $\|\cdot\|^2_{\mathscr{H}}$, implies

$$s(s+2c)\|v^{n+1}\|^2_{\mathscr{H}} - s^2\|v^n\|^2_{\mathscr{H}} + \frac{s^2}{q}\sum_{\ell=1}^{q}\|v_\ell^{n+1} - v^n\|^2_{\mathscr{H}} \leq \frac{2}{q}\sum_{\ell=1}^{q}(-\mathscr{F}_\ell u, s v_\ell^{n+1})_{\mathscr{H}}.$$

As $\mathscr{F}u = \sum_{\ell=1}^q \mathscr{F}_\ell u = 0$, the right-hand side of the above inequality is bounded by

$$\begin{aligned} \frac{2}{q}\sum_{\ell=1}^{q}(-\mathscr{F}_\ell u, s v_\ell^{n+1})_{\mathscr{H}} &= \frac{2}{q}\sum_{\ell=1}^{q}\Big(\big(-\mathscr{F}_\ell u, s(v_\ell^{n+1} - v^n)\big)_{\mathscr{H}} + (-\mathscr{F}_\ell u, s v^n)_{\mathscr{H}}\Big) \\ &= \frac{2}{q}\sum_{\ell=1}^{q}\big(-\mathscr{F}_\ell u, s(v_\ell^{n+1} - v^n)\big)_{\mathscr{H}} \leq \frac{1}{q}\sum_{\ell=1}^{q}\big(\|\mathscr{F}_\ell u\|^2_{\mathscr{H}} + s^2\|v_\ell^{n+1} - v^n\|^2_{\mathscr{H}}\big). \end{aligned}$$

Hence, it follows that

$$s(s+2c)\|v^{n+1}\|^2_{\mathscr{H}} - s^2\|v^n\|^2_{\mathscr{H}} \leq \frac{1}{q}\sum_{\ell=1}^{q}\|\mathscr{F}_\ell u\|^2_{\mathscr{H}}.$$

For the constant $C(\mathscr{F}_\ell) = 1/q\sum_{\ell=1}^q \|\mathscr{F}_\ell u\|_{\mathscr{H}}^2$, we then obtain the bound

$$\begin{aligned}\|u^N - u\|_{\mathscr{H}}^2 &\le \frac{1}{(1+2c/s)^N}\|u^0 - u\|_{\mathscr{H}}^2 + C(\mathscr{F}_\ell)\frac{1}{s^2}\sum_{n=1}^N \frac{1}{(1+2c/s)^n}\\ &= \frac{1}{(1+2c/s)^N}\|u^0 - u\|_{\mathscr{H}}^2 + C(\mathscr{F}_\ell)\frac{1}{2cs}\big(1 - \frac{1}{(1+2c/s)^N}\big).\end{aligned}$$

The parameter choice $s = C\sqrt{N}$ then gives the sought after convergence in $\mathscr{H}$, as $1/(1+C/\sqrt{N})^N \to 0$ as $N \to \infty$. □

6 Degenerate elliptic-parabolic equations with p-structures

Multiplying the degenerate elliptic-parabolic equation (1) with a sufficiently regular test function v, with $v(T) = 0$, formally gives the weak form

$$\int_0^T \int_\Omega -\gamma u \partial_t v + \alpha(t,\nabla u)\cdot\nabla v + \beta(t,u)v + f(t)v \,\mathrm{d}x\mathrm{d}t = 0.$$

Comparing with (9) motivates the choices

$$H = L^2(\Omega), \quad V = W^{1,p}(\Omega), \quad \text{and} \quad (Mu)(x) = \gamma(x)u(x),$$

where $p \in [2,\infty)$, $\gamma \in L^\infty(\Omega)$ is nonnegative, together with the operators $A(t)\colon V \to V^*$, $t \in (0,T)$, given by

$$\langle A(t)u, v\rangle_{V^*\times V} = \int_\Omega \alpha(t,\nabla u)\cdot\nabla v + \beta(t,u)v\,\mathrm{d}x \quad \text{for } u,v \in V.$$

To decompose H we consider $H_\ell = L^2(\Omega_\ell)$, with the usual norm. The connection between H and H_ℓ is the given by the zero extension operator $E_\ell\colon H_\ell \to H$ and the restriction operator $R_\ell\colon H \to H_\ell$, i.e.,

$$(E_\ell u_\ell)(x) = \begin{cases} 0 & \text{for } x \in \Omega\setminus\Omega_\ell \\ u_\ell(x) & \text{for } x \in \Omega_\ell\end{cases} \quad \text{and} \quad R_\ell u = u|_{\Omega_\ell}$$

for all $u_\ell \in H_\ell$ and $u \in H$. Note that these operators fulfill $(E_\ell u_\ell, u)_H = (u_\ell, R_\ell u)_{H_\ell}$.

Next, we will introduce the decomposed operators and source terms. To this end, recall the overlapping subdomains $\{\Omega_\ell\}_{\ell=1}^q$ from Section 2, and consider the partition of unities with the weights $\chi_\ell^\alpha = E_\ell a_\ell$, $\chi_\ell^\beta = E_\ell b_\ell$, and $\chi_\ell^\gamma = E_\ell g_\ell$. Here, we assume

$$a_\ell \in W_0^{1,\infty}(\Omega_\ell), \quad b_\ell \in \{v \in L^\infty(\Omega_\ell) : v(x) \ge c \text{ for a.e. } x \in \Omega_\ell\}, \quad \text{and} \quad g_\ell \in L^\infty(\Omega_\ell),$$

for $\ell = 1,\dots,q$. A possible choice of χ_ℓ^f will be presented in Example 2.

For an arbitrary weight $w_\ell \in L^\infty(\Omega_\ell)$, let $L^p(\Omega_\ell, w_\ell)$ be the set of measurable functions u on Ω_ℓ such that the weighted norm

$$\|u\|_{L^p(\Omega_\ell,w_\ell)} = \big(\int_{\Omega_\ell} w_\ell |u|^p \,\mathrm{d}x\big)^{\frac{1}{p}}$$

is finite. This is a separable and reflexive Banach space. If $w_\ell(x) \geq c$ for a.e. $x \in \Omega_\ell$ then $L^p(\Omega_\ell, w_\ell)$ is isometric isomorphic to $L^p(\Omega_\ell)$. We can then define the spaces

$$V_\ell = \big\{u \in L^p(\Omega_\ell, b_\ell) : \text{ there exists } z_j \in L^p(\Omega_\ell, a_\ell), j = 1, \dots, d, \text{ with}$$
$$\langle a_\ell(\nabla u)_j, v\rangle_{W^{-1,p}(\Omega_\ell)\times W_0^{1,p}(\Omega_\ell)} = -\int_{\Omega_\ell} a_\ell z_j v \, \mathrm{d}x \quad \text{for all } v \in W_0^{1,p}(\Omega_\ell)\big\}$$

for $\ell = 1, \dots, q$. We equip V_ℓ with the norm

$$\|u\|_{V_\ell} = \big(\|\nabla u\|^p_{L^p(\Omega_\ell, a_\ell)^d} + \|u\|^p_{L^p(\Omega_\ell, b_\ell)}\big)^{1/p}.$$

As for standard weak derivatives, we will not distinguish between the distributional gradient of $u \in V_\ell$ and the associated vector of weighted L^p-functions $z = (z_1, \dots z_d)$. Instead, both will be denoted by ∇u.

Lemma 3 *The spaces $V_\ell, \ell = 1, \dots, q$, are separable, reflexive Banach spaces, which are all densely embedded into H_ℓ.*

The proof follows as in [4, Lemmas 1–3] together with the observation that the separability also holds by [1, Theorem 1.22].

The operators $M_\ell \colon H_\ell \to H_\ell, \ell = 1, \dots, q$, given by

$$(M_\ell u)(x) = g_\ell(x)(R_\ell \gamma)(x)u(x) \quad \text{for a.e. } x \in \Omega_\ell,$$

are then linear, bounded, monotone, symmetric, and fulfill $\sum_{\ell=1}^q E_\ell M_\ell R_\ell = M$. The families of operators $A_\ell(t) \colon V_\ell \to V_\ell^*$, $t \in (0,T), \ell = 1, \dots, q$, are defined by

$$\langle A_\ell(t)u, v\rangle_{V_\ell^* \times V_\ell} = \int_{\Omega_\ell} a_\ell \alpha(t, \nabla u) \cdot \nabla v + b_\ell \beta(t, u) v \, \mathrm{d}x \quad \text{for } u, v \in V_\ell.$$

We will consider degenerate elliptic-parabolic equations that have a p-structure, i.e., the functions α, β have the properties below.

Assumption 5 *Let the function $d_1 \in L^{p/(p-1)}\big(\Omega \times (0,T)\big)$ be nonnegative and $d_2 \in L^1\big(\Omega \times (0,T)\big)$. The functions $\alpha \colon \Omega \times (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta \colon \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfy, for a.e. $(x,t) \in \Omega \times (0,T)$,*

1. *the maps $\alpha(x,t,z)$ and $\beta(x,t,y)$ are measurable in x, t, and continuous in y, z;*
2. *for all $y \in \mathbb{R}, z \in \mathbb{R}^d$ one has the bounds*

$$|\alpha(x,t,z)| \leq C|z|^{p-1} + d_1(x,t) \quad \text{and} \quad |\beta(x,t,y)| \leq C|y|^{p-1} + d_1(x,t);$$

3. *for all $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$ it holds that*

$$\big(\alpha(x,t,z_1) - \alpha(x,t,z_2)\big) \cdot (z_1 - z_2) + \big(\beta(x,t,y_1) - \beta(x,t,y_2)\big)(y_1 - y_2)$$
$$\geq c(|z_1 - z_2|^p + |y_1 - y_2|^p);$$

4. *for all $y \in \mathbb{R}, z \in \mathbb{R}^d$ one has $\alpha(x,t,z) \cdot z + \beta(x,t,y)y \geq c(|z|^p + |y|^p) - d_2(x,t)$.*

Example 1 The standard case that fulfills Assumption 5 is the parabolic p-Laplace problem with a nonlinear reaction term, i.e.,

$$\alpha(x,t,z) = |z|^{p-2}z \quad \text{and} \quad \beta(x,t,y) = |y|^{p-2}y + \lambda y,$$

where $\lambda \geq 0$.

Assumption 6 *The source term $f \in \mathscr{V}^*$ in (1) can be decomposed as*

$$\langle f, v\rangle_{\mathscr{V}^* \times \mathscr{V}} = \sum_{\ell=1}^{q} \langle f_\ell, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell}$$

for all $v \in \mathscr{V}$.

Example 2 Let $\eta_0 \in L^{p/(p-1)}\big(\Omega \times (0,T)\big)$ and $\eta \in L^{p/(p-1)}\big(\Omega \times (0,T)\big)^d$. Then the functional f given by

$$\langle f, v\rangle_{\mathscr{V}^* \times \mathscr{V}} = \int_0^T \int_\Omega \eta_0(t)v(t) + \eta(t) \cdot \nabla v(t)\,\mathrm{d}x\mathrm{d}t \quad \text{for } v \in V$$

is an element in $\mathscr{V}^*$. We can then decompose f into elements f_ℓ in $\mathscr{V}_\ell^*$ through

$$\langle f_\ell, v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} = \int_0^T \int_{\Omega_\ell} b_\ell R_\ell \eta_0(t)v(t) + a_\ell R_\ell \eta(t) \cdot \nabla v(t)\,\mathrm{d}x\mathrm{d}t \quad \text{for } v \in \mathscr{V}_\ell.$$

This implies that

$$\begin{aligned}\sum_{\ell=1}^{q} \langle f_\ell, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} &= \sum_{\ell=1}^{q} \int_0^T \int_{\Omega_\ell} b_\ell R_\ell \eta_0(t) R_\ell v(t) + a_\ell R_\ell \eta(t) \cdot \nabla R_\ell v(t)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \sum_{\ell=1}^{q} (E_\ell b_\ell)\eta_0(t)v(t) + (E_\ell a_\ell)\eta(t) \cdot \nabla v(t)\,\mathrm{d}x\mathrm{d}t \\ &= \langle f, v\rangle_{\mathscr{V}^* \times \mathscr{V}},\end{aligned}$$

for all $v \in \mathscr{V}$, i.e., Assumption 6 holds for this family of functionals.

Lemma 4 *If Assumption 5 holds then $A_\ell(t)\colon V_\ell \to V_\ell^*$ satisfies, for a.e. $t \in (0,T)$,*

1. *$\|A_\ell(t)u\|_{V_\ell^*} \leq C\|u\|_{V_\ell}^{p-1} + d_3(t)$ for all $u \in V_\ell$;*
2. *k_ℓ-monotone with $k_\ell u = c\|u\|_{V_\ell}^p$;*
3. *$A_\ell(t)$ is hemicontinuous;*
4. *$\langle A_\ell(t)u, u\rangle_{V_\ell^* \times V_\ell} \geq c\|u\|_{V_\ell}^p - d_4(t)$ for all $u \in V_\ell$;*
5. *$t \mapsto \langle A_\ell(t)u, v\rangle_{V_\ell^* \times V}$ is measurable on $(0,T)$ for all $u, v \in V_\ell$,*

where $d_3 \in L^{p/(p-1)}(0,T)$ is nonnegative and $d_4 \in L^1(0,T)$. The same properties hold for $A(t)$, with V_ℓ replaced by V.

Proof The first assertion is valid, as for all $u,v \in V_\ell$ and a.e. $t \in (0,T)$ one has that

$$
\begin{aligned}
&|\langle A_\ell(t)u,v\rangle_{V_\ell^*\times V_\ell}\\
&\le \Big(C\big(\int_{\Omega_\ell} a_\ell|\nabla u|^p\,\mathrm{d}x\big)^{\frac{p-1}{p}}+\big(\int_{\Omega_\ell} a_\ell d_1(t)^{\frac{p}{p-1}}\,\mathrm{d}x\big)^{\frac{p-1}{p}}\Big)\big(\int_{\Omega_\ell} a_\ell|\nabla v|^p\,\mathrm{d}x\big)^{\frac{1}{p}}\\
&\quad+\Big(C\big(\int_{\Omega_\ell} b_\ell|u|^p\,\mathrm{d}x\big)^{\frac{p-1}{p}}+\big(\int_{\Omega_\ell} b_\ell d_1(t)^{\frac{p}{p-1}}\,\mathrm{d}x\big)^{\frac{p-1}{p}}\Big)\big(\int_{\Omega_\ell} b_\ell|v|^p\,\mathrm{d}x\big)^{\frac{1}{p}}\\
&\le C(\|u\|_{V_\ell}^{p-1}+\|d_1(t)\|_{L^{p/(p-1)}(\Omega)})\|v\|_{V_\ell}.
\end{aligned}
$$

The second assertion holds, as

$$
\begin{aligned}
&\langle A_\ell(t)u-A_\ell(t)v,u-v\rangle_{V_\ell^*\times V_\ell}\\
&\quad=\int_{\Omega_\ell} a_\ell\big(\alpha(t,\nabla u)-\alpha(t,\nabla v)\big)\cdot\nabla(u-v)+b_\ell\big(\beta(t,u)-\beta(t,v)\big)(u-v)\,\mathrm{d}x\\
&\quad\ge c\big(\|\nabla(u-v)\|_{L^p(\Omega_\ell,a_\ell)^d}^p+\|u-v\|_{L^p(\Omega_\ell,b_\ell)}^p\big)=c\|u-v\|_{V_\ell}^p \quad \text{for all } u,v\in V_\ell.
\end{aligned}
$$

To prove the third assertion, consider a sequence $\{\varepsilon_n\}\subset[0,1]$ with the limit ε, elements $u,v,w\in V_\ell$, and the function

$$
h(\varepsilon,x,t)=a_\ell(x)\alpha\big(x,t,(\nabla u+\varepsilon\nabla v)(x)\big)\cdot\nabla w(x)+b_\ell(x)\beta\big(x,t,(u+\varepsilon v)(x)\big)w(x).
$$

By Assumption 5, we have $h(\varepsilon_n,x,t)\to h(\varepsilon,x,t)$ for a.e. $(x,t)\in\Omega_\ell\times(0,T)$, and

$$
\begin{aligned}
|h(\varepsilon,x,t)|\le\ & Ca_\ell(x)\big((|\nabla u(x)|+|\nabla v(x)|)^{p-1}+c|d_1(x,t)|\big)|\nabla w(x)|\\
&+Cb_\ell(x)\big((|u(x)|+|v(x)|)^{p-1}+c|d_1(x,t)|\big)|w(x)|,
\end{aligned}
$$

where the right-hand side is in $L^1(\Omega_\ell)$ for a.e. t. Hence, the dominated convergence theorem gives

$$
\lim_{n\to\infty}\langle A_\ell(t)(u+\varepsilon_n w),v\rangle_{V_\ell^*\times V_\ell}=\lim_{n\to\infty}\int_{\Omega_\ell} h(\varepsilon_n,x,t)\,\mathrm{d}x=\langle A_\ell(t)(u+\varepsilon w),v\rangle_{V_\ell^*\times V_\ell},
$$

which implies that $A_\ell(t)$ is hemicontinuous for a.e. t. The fourth assertion holds, as

$$
\begin{aligned}
\langle A_\ell(t)u,u\rangle_{V_\ell^*\times V_\ell}&=\int_{\Omega_\ell} a_\ell\alpha(t,\nabla u)\cdot\nabla u+b_\ell\beta(t,u)u\,\mathrm{d}x\\
&\ge\int_{\Omega_\ell} c(a_\ell|\nabla u|^p+b_\ell|u|^p)-d_2(x,t)\,\mathrm{d}x\ge c\|u\|_{V_\ell}^p-\|d_2(\cdot,t)\|_{L^1(\Omega_\ell)}
\end{aligned}
$$

for all $u\in V_\ell$. The final assertion holds, by the measurability of α,β and the argument in [22, Section 30.4]. Repeating the same proof, with the weights a_ℓ,b_ℓ replaced by 1, gives the same properties for $A(t)\colon V\to V^*$. □

Lemma 4 together with [22, Section 30.3b] gives that the induced operators $\mathscr{A}_\ell\colon\mathscr{V}_\ell\to\mathscr{V}_\ell^*$, $\ell=1,\dots,q$, are all bounded, hemicontinuous, k_ℓ-monotone, and coercive. Here, the function k_ℓ has the form

$$
k_\ell u=c\|u\|_{\mathscr{V}_\ell}^p.
$$

The same properties also hold for $\mathscr{A}: \mathscr{V} \to \mathscr{V}^*$. That is, in the context of degenerate elliptic-parabolic equations, the problem sets $(V, H, M, \mathscr{A})$ and $(V_\ell, H_\ell, M_\ell, \mathscr{A}_\ell)$ are all proper. Hence, Assumption 5 implies Assumption 1.

Validating Assumption 2 can be done as follows. Since $L^p(\Omega_\ell) \subseteq L^p(\Omega_\ell, w_\ell)$ for every weight function w_ℓ that we have considered, we find that $W^{1,p}(\Omega_\ell) \subset V_\ell$. For $u \in V$, it therefore follows that R_ℓ is bounded from V to V_ℓ as

$$\begin{aligned}\|R_\ell u\|_{V_\ell}^p &= \big\|\nabla R_\ell u\big\|_{L^p(\Omega_\ell,a_\ell)^d}^p + \big\|R_\ell u\big\|_{L^p(\Omega_\ell,b_\ell)}^p \leq C\big(\big\|\nabla R_\ell u\big\|_{L^p(\Omega_\ell)^d}^p + \big\|R_\ell u\big\|_{L^p(\Omega_\ell)}^p\big)\\ &= C\big(\big\|E_\ell \nabla R_\ell u\big\|_{L^p(\Omega)^d}^p + \big\|E_\ell R_\ell u\big\|_{L^p(\Omega)}^p\big) \leq C\|u\|_V^p.\end{aligned}$$

The second part of Assumption 2 follows from

$$\begin{aligned}\|u\|_V^p &= \big\|\nabla u\big\|_{L^p(\Omega)^d}^p + \big\|u\big\|_{L^p(\Omega)}^p\\ &\leq \sum_{\ell=1}^q \big\|\nabla R_\ell u\big\|_{L^p(\Omega_\ell,a_\ell)^d}^p + \sum_{\ell=1}^q \big\|R_\ell u\big\|_{L^p(\Omega_\ell,b_\ell)}^p = \sum_{\ell=1}^q \big\|R_\ell u\big\|_{V_\ell}^p.\end{aligned}$$

To prove that Assumptions 5 and 6 implies Assumption 3 first observe that

$$\begin{aligned}&\sum_{\ell=1}^q \langle \mathscr{A}_\ell \mathscr{R}_\ell u, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell}\\ &= \sum_{\ell=1}^q \int_0^T \int_{\Omega_\ell} a_\ell \alpha\big(t, \nabla R_\ell u(t)\big) \cdot \nabla R_\ell v(t) + b_\ell \beta\big(t, R_\ell u(t)\big) R_\ell v(t)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T \int_\Omega \sum_{\ell=1}^q (E_\ell a_\ell)\alpha\big(t, \nabla u(t)\big) \cdot \nabla v(t) + (E_\ell b_\ell)\beta\big(t, u(t)\big) v(t)\,\mathrm{d}x\mathrm{d}t\\ &= \langle \mathscr{A} u, v\rangle_{\mathscr{V}^* \times \mathscr{V}}\end{aligned}$$

for every $u, v \in \mathscr{V}$. Hence, by Corollary 1, one has the equality

$$\begin{aligned}\Big(\sum_{\ell=1}^q \mathscr{F}_\ell u, v\Big)_{\mathscr{H}} &= \sum_{\ell=1}^q \big((\mathrm{d}/\mathrm{d}t \mathscr{M}_\ell + \mathscr{A}_\ell)\mathscr{R}_\ell u + f_\ell, \mathscr{R}_\ell v\big)_{\mathscr{H}_\ell}\\ &= \sum_{\ell=1}^q \langle \mathrm{d}/\mathrm{d}t \mathscr{M}_\ell \mathscr{R}_\ell u, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} + \langle \mathscr{A}_\ell \mathscr{R}_\ell u, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell} + \langle f_\ell, \mathscr{R}_\ell v\rangle_{\mathscr{V}_\ell^* \times \mathscr{V}_\ell}\\ &= \langle (\mathrm{d}/\mathrm{d}t \mathscr{M} + \mathscr{A})u + f, v\rangle_{\mathscr{V}^* \times \mathscr{V}} = \langle \mathscr{F} u, v\rangle_{\mathscr{V}^* \times \mathscr{V}}\end{aligned}$$

for every $u \in \cap_{\ell=1}^q D(\mathscr{F}_\ell), v \in \mathscr{V}$. As $\mathscr{V}$ is dense in $\mathscr{H}$, we have $u \in D(\mathscr{F})$ and $\sum_{\ell=1}^q \mathscr{F}_\ell u = \mathscr{F} u$ in $\mathscr{H}$, i.e., Assumption 3 holds.

With this setting Theorem 1 translates into the convergence result below.

Corollary 2 *Consider the Peaceman–Rachford (4) or Douglas–Rachford (5) approximation $\{u_1^n, u_2^n\}_{n\in\mathbb{N}}$ of the solution u to the degenerate elliptic-parabolic problem $\mathscr{F} u = 0$. If Assumptions 4 to 6 hold and $u_2^0 \in D(\mathscr{F}_2)$ then*

$$\lim_{n\to\infty} \sum_{\ell=1}^2 c\|\nabla(u_\ell^n - u)\|_{L^p(0,T;L^p(\Omega_\ell,a_\ell)^d)}^p + c\|u_\ell^n - u\|_{L^p(0,T;L^p(\Omega_\ell,b_\ell))}^p = 0$$

for every method parameter $s > 0$.

For the additive splitting (6), one needs a slight modification, as Assumption 5 does not imply the condition $k_\ell \geq c\|\cdot\|^2_{\mathscr{H}}$. To this end, we will consider the case with $\gamma(x) \geq \gamma_0 > 0$ for a.e. $x \in \Omega$, i.e., a degenerate parabolic equation. In this setting, the variable change $\hat{u}(t) = \mathrm{e}^{-qt}u(t)$ gives the "shifted" equation

$$\hat{\mathscr{F}}\hat{u} = \mathrm{d}/\mathrm{d}t\mathscr{M}\hat{u} + \hat{\mathscr{A}}\hat{u} + \hat{f} = 0 \quad \text{in } \mathscr{V}^*,$$

with $\hat{f}(t) = \mathrm{e}^{-qt}f(t)$ and

$$\langle \hat{\mathscr{A}}\hat{u}, v\rangle_{\mathscr{V}^*\times\mathscr{V}} = \int_0^T \int_\Omega \mathrm{e}^{-qt}\alpha(t, \mathrm{e}^{qt}\nabla\hat{u})\cdot\nabla v + \left(\mathrm{e}^{-qt}\beta(t, \mathrm{e}^{qt}\hat{u}) + q\gamma\hat{u}\right)v\,\mathrm{d}x\mathrm{d}t.$$

Note that Assumption 5 still holds for the operator above. This implies that $\hat{\mathscr{F}}\hat{u} = 0$ has a unique solution. Furthermore, consider the decompositions, with $u, v \in \mathscr{V}_\ell$,

$$\langle \hat{\mathscr{A}_\ell}u, v\rangle_{\mathscr{V}_\ell^*\times\mathscr{V}_\ell} = \int_0^T \int_{\Omega_\ell} a_\ell \mathrm{e}^{-tq}\alpha(t, \mathrm{e}^{tq}\nabla u)\cdot\nabla v + \left(b_\ell \mathrm{e}^{-tq}\beta(t, \mathrm{e}^{tq}u) + \gamma u\right)v\,\mathrm{d}x\mathrm{d}t.$$

The same proof as for Lemma 4 gives that $(V_\ell, H_\ell, M_\ell, \hat{\mathscr{A}_\ell})$ is proper. Here, $\hat{\mathscr{A}_\ell}$ is $\hat{k}_\ell$-monotone with

$$\hat{k}_\ell u = c\|\nabla u\|^p_{\mathscr{V}_\ell} + \gamma_0\|u\|^2_{\mathscr{H}}.$$

Applying (6) to this new decomposition will be referred to as the *shifted additive splitting*. Theorem 2 then gives the following convergence result.

Corollary 3 *Consider the shifted additive splitting approximation* $\{\hat{u}^n\}_{n=1}^N$ *of the solution* u *to the degenerate parabolic problem* $\mathscr{F}u = 0$*. If Assumptions 4 to 6 hold and* $\gamma(x) \geq \gamma_0 > 0$ *for a.e.* $x \in \Omega$*, then*

$$\lim_{N\to\infty} \|\mathrm{e}^{qt}\hat{u}^N - u\|_{\mathscr{H}} = 0$$

for the parameter choice $s = C\sqrt{N}$*.*

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