ADAPTIVE VARIATIONAL NONLINEAR CHIRP MODE DECOMPOSITION

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ABSTRACT

Variational nonlinear chirp mode decomposition (VNCMD) is a recently introduced method for nonlinear chirp signal decomposition that has aroused notable attention in various fields. One limiting aspect of the method is that its performance relies heavily on the setting of the bandwidth parameter. To overcome this problem, we here propose a Bayesian implementation of the VNCMD, which can adaptively estimate the instantaneous amplitudes and frequencies of the nonlinear chirp signals, and then learn the active dictionary in a data-driven manner, thereby enabling a high-resolution time-frequency representation. Numerical example of both simulated and measured data illustrate the resulting improvement performance of the proposed method.

***Index Terms*—** Nonlinear chirp signal, mode decomposition, time-frequency analysis, adaptive estimation.

1. INTRODUCTION

Nonlinear chirp signals (NCSs) displaying non-stationary structures have drawn increasing attention in a wide set of applications, ranging from speech analysis to fault diagnosis. For such signals, the traditional methods, such as the periodogram, which only provides limited global frequency information, lack the ability to characterize time-varying modes of the NCSs.

In order to characterize the time-frequency (TF) information of such signals, Gabor proposed the short-time Fourier transform (STFT) [1]. The method performs a joint analysis over time and frequency, describing the characteristics of the NCSs using a time-frequency representation (TFR). However, this approach is well-known to suffer from having low-resolution due to Heisenberg's uncertainty principle. To alleviate this problem, notable efforts have focused on the improvement of the TFR resolution. For instance, the synchrosqueezed transform (SST) utilized a post-processing technique to search the TF coefficient of each mode [2]. In order to generalize the method to non-negligible frequency modulations, Oberlin and Meignen proposed the second-order SST (SST2) by introducing high-order approximation operators [3]. Regrettably, these methods rely on further steps, e.g., the ridge detection algorithm, and fail to form accurate estimates of strongly time-varying NCSs.

Employing a data-driven recursive sifting procedure, Huang *et al.* proposed the empirical mode decomposition (EMD) [4], being able to adaptively extract a set of oscillation modes. However, the algorithm experiences some problems, including but not limited to the lack of mathematical basis and being very sensitive to noise. To overcome these problems, a series of EMD-based methods have been proposed [5, 6, 7]. Later, exploiting a Wiener filter bank, Dragomiretskiy and Zosso proposed the variational mode decomposition (VMD) to demodulate NCSs employing convex optimization theory [8]. However, these methods are unsuitable for wide-band signals. More recently, Chen *et al.* proposed the variational nonlinear chirp mode decomposition (VNCMD) [9], converting time-varying NCSs into narrow-band signals to directly estimate the instantaneous amplitudes (IAs) and instantaneous frequencies (IFs). This method offers preferable performance, although its performance relies heavily on the setting of the user-selected bandwidth parameter, α.

In this paper, we consider the mode decomposition problem from a Bayesian perspective. Specifically, we propose an adaptive implementation of the VNCMD, termed the AVNCMD, which can adaptively estimate the IAs and IFs, and forming the dictionary in a data-driven manner, thereby constructing a high-resolution TFR. In addition, rather than providing a point estimate for each parameter, a full posterior density function is inferred, which may then be used to give a sense of confidence for the estimator. Numerical simulations illustrate the improvement brought about by using the proposed method on both simulated and measured data.

The rest of the paper is structured as follows: Section 2 provides the theoretical background and a brief description of the VNCMD. In Section 3, we consider the mode decomposition problem from a Bayesian perspective, and present further details of the AVNCMD algorithm. Numerical results are provided in Section 4. Finally, our conclusions are summarized in Section 5.

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2. BACKGROUND

2.1. Nonlinear Chirp Signal Model

Let $g(t)$ denote a NCS consisting of K modes, such that (see, e.g. [9, 10])

$$g(t)=\sum_{k=1}^{K} g_k(t)=\sum_{k=1}^{K} a_k(t)\cos\left(2\pi\int_0^t f_k(s)ds+\phi_k\right), \tag{1}$$

where $a_k(t)$ and $f_k(t)$ denote the non-negative IA and IF of the k-th mode, respectively, with ϕ_k denoting the initial phase. Then, (1) may be rewritten into a demodulated form as

$$g(t)=\sum_{k=1}^{K} u_k(t)\cos\left(2\pi\int_0^t \tilde{f}_k(s)ds\right)+v_k(t)\sin\left(2\pi\int_0^t \tilde{f}_k(s)ds\right), \tag{2}$$

where $\tilde{f}_k(t)$ denotes the frequency function of the demodulate operators, with the two demodulated signals

$$u_k(t)=a_k(t)\cos\left(2\pi\int_0^t\left(f_k(s)-\tilde{f}_k(s)\right)ds+\phi_k\right), \tag{3}$$

$$v_k(t)=-a_k(t)\sin\left(2\pi\int_0^t\left(f_k(s)-\tilde{f}_k(s)\right)ds+\phi_k\right). \tag{4}$$

2.2. Variational Nonlinear Chirp Mode Decomposition

The VNCMD [9] is a generalization of the VMD [8] to analyze wide-band signals effectively. The main idea of VNCMD is to minmize the bandwidth of the two demodulated signals, $u_k(t)$ and $v_k(t)$, simultaneously. Specifically, the method is focused on solving the optimization problem

$$\min_{u_k,v_k,\tilde{f}_k}\ \alpha||g(t)-\sum_{k=1}^{K} g_k(t)||_2^2+\sum_{k=1}^{K}\left(\left\|u_k''(t)\right\|_2^2+\left\|v_k''(t)\right\|_2^2\right) \tag{5}$$

which, together with the augmented Lagrangian multiplier and the alternating direction multiplier method are adopted to solve (5). The details are available in [9]. The bandwidth parameter, α, in (5) is a user-defined constant that needs to be set in advance, which often results in it being unable to adapt to different NCSs, occationally leading to unsatisfactory demodulation results. For most NCSs, it is difficult to determine a suitable value for α without further prior knowledge.

3. PROPOSED METHOD

3.1. Estimating the Nonlinear Chirp Signal

First, supposing the signal is sampled at $t=t_1,t_2,\ldots,t_n$, the optimization problem (5) may be simplified as

$$\min_{\boldsymbol{x}}\ \alpha||\boldsymbol{g}-\boldsymbol{A}\boldsymbol{x}||_2^2+||\boldsymbol{D}\boldsymbol{x}||_2^2, \tag{6}$$

where $\boldsymbol{g}=[\,g(t_1)\ \ \ldots\ \ g(t_n)\,]^T$, $\boldsymbol{x}=[\,\boldsymbol{x}_1^T\ \ \ldots\ \ \boldsymbol{x}_K^T\,]^T$, $\boldsymbol{x}_k=[\,\boldsymbol{u}_k^T\ \ \boldsymbol{v}_k^T\,]^T$, $\boldsymbol{A}=[\,\boldsymbol{A}_1\ \ \ldots\ \ \boldsymbol{A}_K\,]$, $\boldsymbol{A}_k=[\,\boldsymbol{C}_k\ \ \boldsymbol{S}_k\,]$, $\boldsymbol{u}_k=[\,u_k(t_1)\ \ \ldots\ \ u_k(t_n)\,]^T$, $\boldsymbol{v}_k=[\,v_k(t_1)\ \ \ldots\ \ v_k(t_n)\,]^T$, $\boldsymbol{C}_k=diag\left[\ \cos(\theta_k(t_1))\ \ \ldots\ \ \cos(\theta_k(t_n))\ \right]$, $\boldsymbol{S}_k=diag\left[\ \sin(\theta_k(t_1))\ \ \ldots\ \ \sin(\theta_k(t_n))\ \right]$, $\theta_k(t)=2\pi\int_0^t\tilde{f}_k(s)ds$, and $\boldsymbol{D}=diag\left[\ \boldsymbol{H}\ \ \ldots\ \ \boldsymbol{H}\ \right]$ ($2K$ block matrix), with $\boldsymbol{H}$ denoting the second-order difference matrix, i.e.,

$$\boldsymbol{H}=\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & -2 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & -2 & 1 & 0\\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{bmatrix}\in R^{(n-2)\times n}.$$

Since $\boldsymbol{D}$ has full row rank, one may, reminiscent to [11], construct a full rank matrix $\widetilde{\boldsymbol{D}}=\begin{bmatrix}\boldsymbol{D}\\ \boldsymbol{M}\end{bmatrix}$ by finding a matrix $\boldsymbol{M}$ whose rows are orthogonal to those in $\boldsymbol{D}$. Then, introduce the variable $\boldsymbol{z}=\begin{bmatrix}\boldsymbol{w}\\ \boldsymbol{\theta}\end{bmatrix}=\widetilde{\boldsymbol{D}}\boldsymbol{x}$, yielding the equivalent minimization problem

$$\min_{\boldsymbol{z}}\ \alpha||\boldsymbol{g}-\boldsymbol{A}\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{z}||_2^2+||\boldsymbol{w}||_2^2. \tag{7}$$

We then write $\boldsymbol{A}\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{z}=\boldsymbol{B}_1\boldsymbol{w}+\boldsymbol{B}_2\boldsymbol{\theta}$. Thus, the solution of (7) with respect to $\boldsymbol{\theta}$ is given by a linear regression, i.e.,

$$\hat{\boldsymbol{\theta}}=\left(\boldsymbol{B}_2^T\boldsymbol{B}_2\right)^{-1}\boldsymbol{B}_2^T\left(\boldsymbol{g}-\boldsymbol{B}_1\hat{\boldsymbol{w}}\right). \tag{8}$$

Substituting (8) into (7), the estimation of $\boldsymbol{w}$ may be formulated as

$$\min_{\boldsymbol{w}}\ \alpha||\boldsymbol{y}-\boldsymbol{\Phi}\boldsymbol{w}||_2^2+||\boldsymbol{w}||_2^2, \tag{9}$$

where $\boldsymbol{y}=(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{g}$, $\boldsymbol{\Phi}=(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{B}_1$, with $\boldsymbol{I}$ denoting the identity matrix, and $\boldsymbol{P}=\boldsymbol{B}_2(\boldsymbol{B}_2^T\boldsymbol{B}_2)^{-1}\boldsymbol{B}_2^T$. As shown below, this optimization may be formed adaptively, after which one may form the solution of (6) as $\hat{\boldsymbol{x}}=\widetilde{\boldsymbol{D}}^{-1}\hat{\boldsymbol{z}}$.

3.2. Bayesian Strategy

In Bayesian modeling, all unknowns are treated as random variables with a specified probability distribution. Assuming that the noise in (9) is independent circular symmetric zero-mean Gaussian noise implies that

$$\mathrm{p}(\boldsymbol{y}\mid\boldsymbol{w},\gamma_0)=\mathcal{N}\left(\boldsymbol{y}\mid\boldsymbol{\Phi}\boldsymbol{w},\gamma_0^{-1}\boldsymbol{I}\right), \tag{10}$$

where γ_0 is the noise precision (inverse noise variance). To enforce the smoothness, the prior for $\boldsymbol{w}$ takes the form[1]

$$\mathrm{p}(\boldsymbol{w}\mid\boldsymbol{\gamma})=\prod_{j=1}^{m}\mathcal{N}\left(w_j\mid 0,\gamma_j^{-1}\right), \tag{11}$$

where $\boldsymbol{\gamma}=[\,\gamma_1\ \ \ldots\ \ \gamma_m\,]^T$, with γ_j denoting the precision of the corresponding parameter w_j.

[1]For convenient, the dimension of $\boldsymbol{w}$ is written as $m\times 1$.

Since (11) is a conjugate prior of (10), the posterior density function for $\boldsymbol{w}$ may be found to be a multivariate Gaussian distribution [12]

$$\mathrm{p}(\boldsymbol{w} \mid \boldsymbol{y}, \gamma_0, \boldsymbol{\gamma}) = \mathcal{N}(\boldsymbol{w} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{12}$$

where

$$\boldsymbol{\mu} = \gamma_0^{-1}\boldsymbol{\Sigma}\boldsymbol{\Phi}^T\boldsymbol{y}, \tag{13}$$
$$\boldsymbol{\Sigma} = (\gamma_0^{-1}\boldsymbol{\Phi}^T\boldsymbol{\Phi} + \boldsymbol{\Lambda})^{-1}, \tag{14}$$

and $\boldsymbol{\Lambda} = diag[\, \gamma_1 \ \ \ldots \ \ \gamma_m \,]$. As in [13], the empirical Bayesian estimation for the hyperparameters, γ_0 and $\boldsymbol{\gamma}$, may be determined by maximizing the marginal likelihood, i.e.,

$$\begin{aligned} \mathcal{L}(\gamma_0, \boldsymbol{\gamma}) &= \ln \mathrm{p}(\boldsymbol{w} \mid \gamma_0, \boldsymbol{\gamma}) \\ &= -\frac{1}{2}\left[m \ln(2\pi) + \ln|\boldsymbol{C}| + \boldsymbol{y}^T\boldsymbol{C}^{-1}\boldsymbol{y}\right], \end{aligned} \tag{15}$$

with $\boldsymbol{C} = \gamma_0^{-1}\boldsymbol{I} + \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Phi}^T$.

In the type-II maximum-likelihood (ML) procedure, the expectation-maximum (EM) algorithm may be used to update the hyperparameters [14, 15], to yield

$$\gamma_j^{(\text{new})} = \frac{1 - \gamma_j^{(\text{old})}\Sigma_{jj}}{\mu_j^2}, \quad j = 1, \ldots, m, \tag{16}$$

$$\gamma_0^{(\text{new})} = \frac{m - \sum_j \gamma_j^{(\text{new})}}{\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\mu}\|^2}. \tag{17}$$

This therefore suggests an iterative procedure, which alternately re-estimates the posterior mean and covariance, using (13) and (14), and then re-estimates the hyperparameters, using (16) and (17), until a suitable convergence criterion (e.g. the update of $\boldsymbol{\mu}$ is negligible) is satisfied. The convergence rate may be accelerated by using numerically efficient implementation techniques [16, 17, 18].

3.3. Data-Driven Implementation

The signal demodulation of the proposed method is completed in an iterative approach. In the i-th iteration, after obtaining $\boldsymbol{x}^{(i)}$, the k-th mode may be estimated as $\boldsymbol{g}_k^{(i)} = \boldsymbol{A}_k^{(i)}\boldsymbol{x}_k^{(i)}$. Next, exploiting the IF information contained in (3) and (4), the arctangent demodulation technique [19, 20] may be introduced to update the IF increment, i.e.,

$$\begin{aligned} \Delta f_k^{(i)}(t) &= -\frac{1}{2\pi}\frac{d}{dt}\arctan\left(\frac{v_k^{(i)}(t)}{u_k^{(i)}(t)}\right) \\ &= \frac{v_k^{(i)}(t)\cdot\left(u_k^{(i)}(t)\right)' - u_i^{k+1}(t)\cdot\left(v_k^{(i)}(t)\right)'}{2\pi\left[\left(u_k^{(i)}(t)\right)^2 + \left(v_k^{(i)}(t)\right)^2\right]}. \end{aligned} \tag{18}$$

Typically, the IF is a smooth function [21]. To avoid the numerical errors caused by discrete time sampling, the IF increment may be corrected by a low-pass filter [9, 10]

$$\Delta\tilde{\boldsymbol{f}}_k^{(i)} = \left(\boldsymbol{I} + \beta\boldsymbol{H}^T\boldsymbol{H}\right)^{-1}\Delta\boldsymbol{f}_k^{(i)}, \tag{19}$$

Table 1. The REs corresponding to the different methods.

	VNCMD			AVNCMD
	$\alpha = 10^{-6}$	$\alpha = 10^{-5}$	$\alpha = 10^{-4}$	-
$g_1(t)$	0.2570	0.0309	0.0904	**0.0079**
$g_2(t)$	0.5994	0.0308	0.0896	**0.0074**
$f_1(t)$	0.0163	0.0042	0.0108	**0.0026**
$f_2(t)$	0.0377	0.0075	0.0099	**0.0027**

where $\Delta\tilde{\boldsymbol{f}}_k^{(i)} = [\,\Delta\tilde{f}_k^{(i)}(t_1) \ \ \ldots \ \ \Delta\tilde{f}_k^{(i)}(t_n)\,]^T$, whereas $\Delta\boldsymbol{f}_k^{(i)} = [\,\Delta f_k^{(i)}(t_1) \ \ \ldots \ \ \Delta f_k^{(i)}(t_n)\,]^T$, with β denoting the filter parameter. Subsequently, the IF is updated as $\tilde{\boldsymbol{f}}_k^{(i+1)} = \tilde{\boldsymbol{f}}_k^{(i)} + \Delta\tilde{\boldsymbol{f}}_k^{(i)}$. Also, exploting the IA information contained in (3) and (4), yields $\boldsymbol{a}_k^{(i)}(t) = \sqrt{\boldsymbol{u}_k^{(i)}(t)^2 + \boldsymbol{v}_k^{(i)}(t)^2}$. Eventually, the dictionary for the next iteration is constructed by utilizing the relationship between $\tilde{\boldsymbol{f}}_k^{(i+1)}$ and $\boldsymbol{A}_k^{(i+1)}$ as detailed above. The above process may be repeated until the update of modes is negligible, forming the AVNCMD algorithm[2]. Notably, the proposed AVNCMD is able to estimate the IAs and IFs of modes, and thus a TFR of NCS can be constructed successfully.

4. NUMERICAL RESULTS

In this section, we test the performance of the proposed AVNCMD on both simulated and measured signals, with comparisons (when appropriate) made to VNCMD [9], STFT [1], SST [2], and SST2 [3].

4.1. Simulated Signal

As the first experiment, we consider a two-component simulated NCS, which may be expressed as

$$g(t) = g_1(t) + g_2(t), \tag{20}$$

where

$$\begin{cases} g_1(t) = (1+0.5\cos(2\pi t))\cos(2\pi(100t+150t^2+\sin(9\pi t))), \\ g_2(t) = (1-0.5\cos(2\pi t))\cos(2\pi(400t-150t^2+\sin(9\pi t))), \end{cases}$$

with the IFs being $f_1(t) = 100 + 300t + 9\pi\cos(9\pi t)$ and $f_2(t) = 400 - 300t + 9\pi\cos(9\pi t)$, respectively. The sampling frequency is set to 2000 Hz, with a sampling period of 1 s.

As a performance measure, the estimated relative error (RE) is employed, i.e., $\mathrm{RE} = \frac{||\hat{\boldsymbol{e}} - \boldsymbol{e}||_2}{||\boldsymbol{e}||_2}$, where $\hat{\boldsymbol{e}}$ and $\boldsymbol{e}$ denote the estimated and theoretical value, respectively. The REs of the estimated modes and IFs are listed in Table 1. As may be seen, the proposed algorithm provides more accurate results with smaller REs. The estimated IFs are shown in Fig.1, clearly illustrating that the proposed method's estimates well match the theoretical values.

[2]The codes are available at https://github.com/HauLiang/AVNCMD.

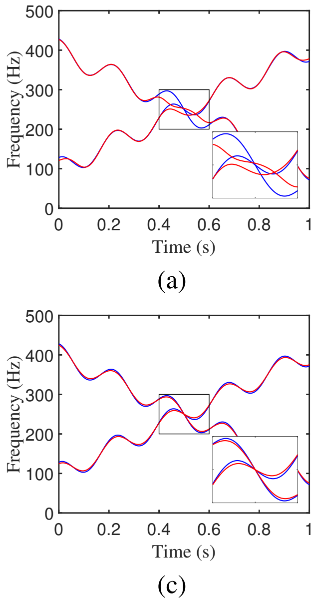

Fig. 1. The estimated IFs by the different algorithms (blue: true; red: estimated): (a) VNCMD ($\alpha = 10^{-6}$), (b) VNCMD ($\alpha = 10^{-5}$), (c) VNCMD ($\alpha = 10^{-4}$), and (d) AVNCMD.

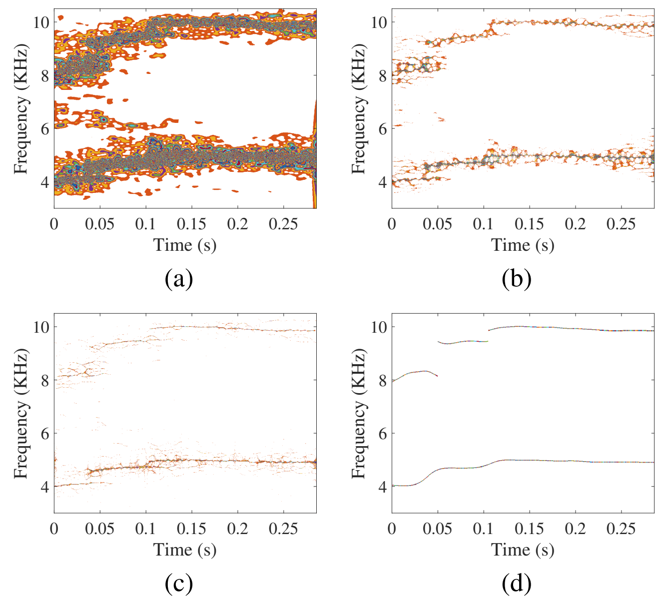

Fig. 2. The TFR results by the different algorithms: (a) STFT, (b) SST, (c) SST2, and (d) AVNCMD.

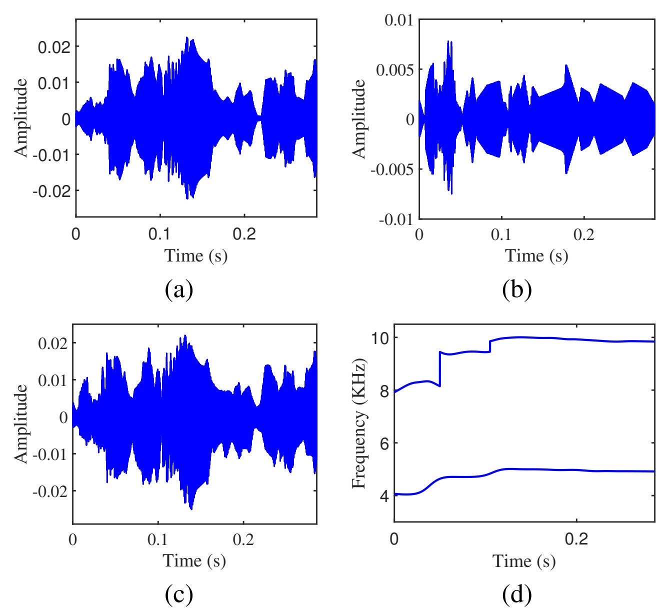

Fig. 3. Demodulated results for a baiji's whistle: (a) estimated (low frequency) mode, (b) estimated (high frequency) mode, (c) sum of the estimated modes, and (d) estimated IFs.

4.2. Real-Life Signal

The proposed approach is also evaluated for real-life signal from the whistle of a baiji, a Yangtze river dolphin (available in [22]). Due to their poor vision and the murky waters of the Yangtze river, the baiji relies on sound for communication, orientation, and feeding. In this experiment, the sampling frequency is 21 kHz.

Fig.2 provides the TFR results by the discussed methods. As may be seen, the STFT is seriously contaminated by environment noise, and suffers from the poor resolution, while the SST and its second-order variant yield limited information, unable to markedly express the signal characteristic. As noted, the above methods fail to directly demodulate the signal modes, and further steps (e.g. the ridge detection algorithm) are required. Conversely, our method yields a high-resolution TFR, being capable of representing the two modes and time-varying features of the signal. The demodulated results by the proposed method are shown in Fig.3. This example indicates the potential of the proposed method in analyzing ocean signals.

5. CONCLUSION

In this paper, we have presented an efficient algorithm, termed the AVNCMD, consisting of several improvements of the VNCMD. Not only can this method demodulate NCSs adaptively, but also estimate their IAs and IFs effectively. Contrary to VNCMD which requires manual setting of parameters, the parameters of the proposed method can be updated automatically. The empirical Bayesian framework provides a parameter-free implementation, using an arctangent demodulation technique to form a data-driven dictionary. The simulation results indicate that the method is more effective than the state-of-the-art algorithms. Finally, we present an application of the method in analyzing the sound of a baiji's whistle.

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