Example: Filtering with disturbance

A common model is

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du + Z(t),$$

where $X(t)$, $t \in \mathbb{R}$, is the input process and $Z(t)$, $t \in \mathbb{R}$, the disturbance process. The processes $X(t)$ and $Z(t)$ are stationary and uncorrelated.

First, for the model above, find the general expressions of the cross-covariance, $r_{X,Y}(\tau)$ and the cross-spectrum, $R_{X,Y}(f)$. Then, also compute the coherence spectrum expressed in $H(f)$, $R_X(f)$ and $R_Z(f)$ using the definition

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)}.$$

Solution: Filtering with disturbance

We find the cross-covariance,

$$\begin{aligned} r_{X,Y}(\tau) &= C[X(t), \int_{-\infty}^{\infty} h(u)X(t+\tau-u)du + Z(t+\tau)], \\ &= \int_{-\infty}^{\infty} h(u)C[X(t), X(t+\tau-u)]du + C[X(t), Z(t+\tau)], \\ &= \int_{-\infty}^{\infty} h(u)r_X(\tau-u)du, \end{aligned}$$

where $C[X(t), Z(t+\tau)]$ is zero as $X(t)$ and $Z(t)$ are uncorrelated.

Solution: Filtering with disturbance

The cross-spectrum follows from above as the Fourier transform of the cross-covariance is

$$R_{X,Y}(f) \quad = \quad \int_{-\infty}^{\infty} r_{X,Y}(\tau) e^{-i2\pi f \tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u) r_X(\tau - u) du \; e^{-i2\pi f \tau} d\tau.$$

Reordering the integrals,

$$\begin{aligned} R_{X,Y}(f) \quad &= \quad \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} r_X(\tau - u) e^{-i2\pi f \tau} d\tau \, du, \\ &= \quad \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} r_X(v) e^{-i2\pi f (v+u)} dv \, du, \end{aligned}$$

where $v = \tau - u$. The cross-spectrum simplifies to

$$R_{X,Y}(f) \quad = \quad \int_{-\infty}^{\infty} h(u) R_X(f) e^{-i2\pi f u} \, du = R_X(f) \int_{-\infty}^{\infty} h(u) e^{-i2\pi f u} \, du,$$

which is

$$R_{X,Y}(f) = H(f) R_X(f).$$

Solution: Coherence spectrum

We find the output spectral density from the covariance function

$$\begin{aligned} r_Y(\tau) &= C[\int_{-\infty}^{\infty} h(u)X(t-u)du + Z(t), \int_{-\infty}^{\infty} h(v)X(t+\tau-v)dv + Z(t+\tau)], \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)r_X(\tau+u-v)dudv + r_Z(\tau) \end{aligned}$$

as $X(t)$ and $Z(t)$ are uncorrelated. The spectral density of the first term is found from the general definition of output spectral density of filtering, (see lecture 7) and the whole expression is

$$R_Y(f) = |H(f)|^2 R_X(f) + R_Z(f).$$

The coherence spectrum is accordingly

$$\kappa^2_{X,Y}(f) = \frac{|H(f)R_X(f)|^2}{R_X(f)(|H(f)|^2 R_X(f) + R_Z(f))} = \frac{|H(f)|^2 R_X(f)}{|H(f)|^2 R_X(f) + R_Z(f)}.$$

Example: Coherence spectrum

Illustration for $R_X(f) = 1$, $R_Z(f) = \sigma^2$ and a lowpass-filter $H(f)$. Note that the coherence spectrum always is one for all frequencies in the undisturbed case, i.e. $\sigma = 0$. This is also easily seen from the the expression for $\kappa^2_{X,Y}(f)$ when $R_Z(f) = 0$.