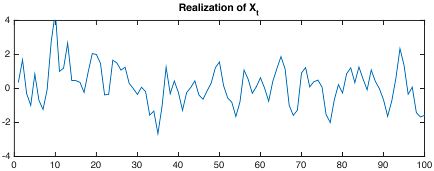
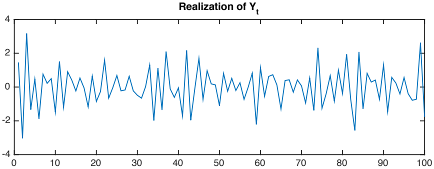
Example smoothing and filtering

The following exercise is a simplified illustration of the similarity between smoothing and filtering. Let X_t and Y_t be zero-mean discrete time stationary processes. The covariance functions are $r_X(0) = 1$, $r_X(\pm 1) = 1/2$ and $r_Y(0) = 1$, $r_Y(\pm 1) = -1/2$. Examples of realizations of the two different processes are seen below.





Example smoothing and filtering

A causal moving average filter with coefficients

$$h(0) = 1/2, \quad \text{and} \quad h(1) = 1/2,$$

is used to filter the two processes giving two new processes Z_t and W_t. The filter $h(t)$ is a lowpass filter, a filter that passes low frequencies and reduces high frequencies, also referred to as a smoothing filter. Studying the realizations on the previous slide we could, without any calculations, argue that the process W_t with input Y_t is the process with most reduced variance after the filtering. The reason is that this is a process where the variance mainly is caused by high frequency content compared with the process X_t which is of mote low frequency content, and therefore a lowpass filter will reduce the variance more for the process Y_t.

Verify, by computing the frequency function $H(f)$ and the amplitude function $|H(f)|$ and calculate the variances, $r_Z(0)$ and $r_W(0)$.

Solution

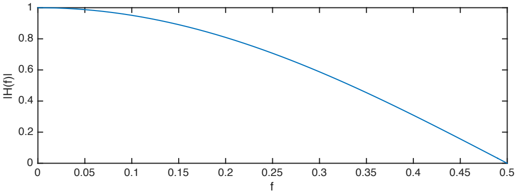
The frequency function is

$$H(f) = \sum_{t=-\infty}^{\infty} h(t)e^{-i2\pi ft} = h(0)e^{-i2\pi f0} + h(1)e^{-i2\pi f1} = \frac{1}{2} + \frac{1}{2}e^{-i2\pi f}.$$

The amplitude function is

$$|H(f)| = \sqrt{H(f)H^*(f)} = \sqrt{(\frac{1}{2} + \frac{1}{2}e^{-i2\pi f})(\frac{1}{2} + \frac{1}{2}e^{i2\pi f})} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos(2\pi f)},$$

which is depicted in the figure below. Low frequencies have an amplification close to one (passed) where high frequencies have lower amplification (reduced).



Solution

We can compute the variance from

$$r(0) = \int_{-1/2}^{1/2} |H(f)|^2 R(f) df.$$

In this case where there are few coefficients in both the impulse response and the covariance functions the variance can easily be calculated from

$$r(\tau) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u)h(v)r(\tau + u - v),$$

with $\tau = 0$ as

$$r(0) = h(0)h(0)r(0) + h(1)h(0)r(1) + h(0)h(1)r(-1) + h(1)h(1)r(0).$$

Solution

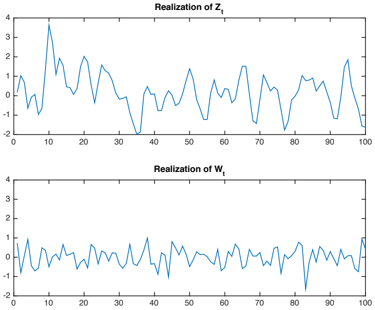
For the two cases the expression is simplified into,

$$r_Z(0) = \frac{1}{2}(r_X(0) + r_X(1)) = \frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4},$$

and

$$r_W(0) = \frac{1}{2}(r_Y(0) + r_Y(1)) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}.$$

Accordingly, the variance of W_t is smaller than for Z_t, which is also shown in the figure below of the output realizations.



Solution

We also study the input spectral densities $R_X(f)$ and $R_Y(f)$ in the upper figure below. The process X_t has more low frequencies and Y_t consists of more high frequencies. The output spectral densities $R_Z(f) = |H(f)|^2 R_X(f)$ and $R_W(f) = |H(f)|^2 R_Y(f)$ are depicted in the lower figure where the low frequency content of X_t is not affected severely in Z_t. The high frequency content of Y_t however is reduced by the lowpass filter and the total spectral density of W_t and the corresponding variance is accordingly very low after the filtering.