Fourier transforms

$g(\tau) \quad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} g(\tau)\, d\tau$
$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha}\, e^{-2\pi\alpha\lvert f\rvert}$

Example: Calculate the corresponding covariance function of the spectral density

$$R(f) = \frac{1}{\beta^2 + f^2}.$$

Solution 1: To adapt to the first Fourier transform pair in the table of formulas:

$e^{-\alpha\lvert\tau\rvert}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$

the spectral density can be rewritten as

$$R(f) = \frac{1}{\beta^2 + f^2} = \frac{(2\pi)^2}{(2\pi\beta)^2 + (2\pi f)^2} = \frac{\pi}{\beta} \frac{2 \cdot 2\pi\beta}{(2\pi\beta)^2 + (2\pi f)^2}.$$

Identifying $2\pi\beta = \alpha$ and a the constant π/β, the corresponding covariance function is given as

$$r(\tau) = \frac{\pi}{\beta} e^{-2\pi\beta\lvert\tau\rvert}.$$

Solution 2: Using the positive symmetry properties of the covariance function and spectral density,

$$R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f\tau} d\tau = 2\int_0^{\infty} r(\tau)\cos(2\pi f\tau) d\tau, \quad r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f\tau} df = 2\int_0^{\infty} R(f)\cos(2\pi f\tau) df,$$

allow us to use the table of Fourier transforms more extensively. From the table of formulas we find

$\frac{1}{\alpha^2+\tau^2}$	$\frac{\pi}{\alpha} e^{-2\pi\alpha\lvert f\rvert}$

where we switch τ for f, resulting in

$\frac{\pi}{\alpha} e^{-2\pi\alpha\lvert\tau\rvert}$	$\frac{1}{\alpha^2+f^2}$.

Replacing $\alpha = \beta$, the solution is $r(\tau) = \frac{\pi}{\beta} e^{-2\pi\beta\lvert\tau\rvert}$ without any calculations at all.