Example 2.5 (modified)

From a stationary stochastic process U_t, for $t = 0, \pm 1, \pm 2, \dots$ of independent variables we construct a new stationary stochastic process X_t by

$$X_t = U_t - 0.5U_{t-1}, \quad t = 0, \pm 1, \pm 2, \dots.$$

Calculate the mean value m_X and covariance function $r_X(\tau)$ for $\tau = 0, \pm 1, \pm 2, \dots$ if U_t has the mean value, $m_U = m$ and variance $r_U(0) = \sigma^2$.

Example 2.5

A stationary stochastic process U_t of independent variables is defined by a variance $V[U_t] = C[U_t, U_t] = r_U(0)$ and all covariances

$$r_U(\tau) = C[U_{t-\tau}, U_t] = 0, \quad \tau = \pm 1, \pm 2, \ldots .$$

Another name is a white noise process .

Example 2.5)

The mean value of X_t, $t = 0, \pm 1, \pm 2, \dots$ is

$$m_X = E[X_t] = E[U_t - 0.5U_{t-1}] = E[U_t] - 0.5E[U_{t-1}],$$

using the rules for expected value of stochastic variables (see the formula booklet). With $m_U = E[U_t] = m$ for all values of t

$$m_X = m_U - 0.5m_U = 0.5m.$$

Example 2.5

The variance of $X_t,\ t = 0, \pm 1, \pm 2, \ldots$ will become

$$\begin{aligned} r_X(0) &= V[X_t] = C[X_t, X_t] = \\ &= C[U_t - 0.5U_{t-1}, U_t - 0.5U_{t-1}] = \\ &= V[U_t] - 0.5C[U_t, U_{t-1}] - 0.5C[U_{t-1}, U_t] + 0.25V[U_{t-1}], \end{aligned}$$

using the rules for variance of stochastic variables. We identify

$$V[U_t] = V[U_{t-1}] = r_U(0),$$

and

$$C[U_t, U_{t-1}] = r_U(-1), \quad C[U_{t-1}, U_t] = r_U(1).$$

For the white noise process, (independent stochastic variables), $r_U(1) = r_U(-1) = 0$. From this follows

$$r_X(0) = r_U(0) + 0.25r_U(0) = 1.25r_U(0) = 1.25\sigma^2.$$

Example 2.5

The covariances of X_t will be as follows:

$$r_X(1) = C[X_{t-1}, X_t] = C[U_{t-1} - 0.5U_{t-2}, U_t - 0.5U_{t-1}] =$$
$$C[U_{t-1}, U_t] - 0.5V[U_{t-1}] - 0.5C[U_{t-2}, U_t] + 0.25C[U_{t-2}, U_{t-1}] =$$
$$r_U(1) - 0.5r_U(0) - 0.5r_U(2) + 0.25r_U(1) = -0.5r_U(0) = -0.5\sigma^2,$$

where most terms are zero according to definition of $r_U(\tau)$.
Similarly we find

$$r_X(2) = C[X_{t-2}, X_t] = C[U_{t-2} - 0.5U_{t-3}, U_t - 0.5U_{t-1}] =$$
$$C[U_{t-2}, U_t] - 0.5C[U_{t-2}, U_{t-1}] - 0.5C[U_{t-3}, U_t] + 0.25C[U_{t-3}, U_{t-1}] =$$
$$r_U(2) - 0.5r_U(1) - 0.5r_U(3) + 0.25r_U(2) = 0.$$

Following the same procedure for other values of τ we find

$$r_X(\tau) = 0, \quad \tau \geq 2.$$

Example 2.5

Next we can use the general rule for real-valued stochastic variables:

$$\begin{aligned} r_X(-\tau) &= C[X_t, X_{t-\tau}] = E[X_t X_{t-\tau}] - m_X^2 = \\ &= E[X_{t-\tau} X_t] - m_X^2 = C[X_{t-\tau}, X_t] = r_X(\tau). \end{aligned}$$

We therefore easily find

$$r_X(-1) = -0.5\sigma^2,$$

and

$$r_X(\tau) = 0, \quad |\tau| \geq 2.$$

Example 2.5-Answer

The expected value is

$$m_X = 0.5m,$$

and the covariance function

$$\begin{aligned} r_X(\tau) = & \quad 1.25\sigma^2, & \tau = 0 \\ = & \; -0.5\sigma^2, & \tau = \pm 1 \\ = & \quad 0, & |\tau| \geq 2. \end{aligned}$$