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# Multilevel Monte Carlo methods for inference in multivariate diffusions

## Bayes @ Lund

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Centre for Mathematical Sciences


# Outline - Title in reverse order

Introduction - Diffusion processes

Multilevel Monte Carlo

Analysis

Simulation study

References

# Bayesian angle

- Model selection is (increasingly) difficult!
- Introduce priors on models, not only parameters!

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- Introduce priors on models, not only parameters!

**My view is that we should focus on physically plausible models, rather than considering all possible black box models!**

Prior information is often codes through differential relations classically described through ODEs and/or PDEs. Examples where stochastic generalizations have been successful includes

- Ebola outbreak - generalized SIR model, King et al. (2015)
- Energy efficiency in houses - thermodynamics, Bacher et al. (2013)
- Spread of Malaria, Bhadra et al. (2011)
- Wind speed forecasting - physics, Iversen et al. (2016)
- Glucose-insulin-glucagon pharmacodynamics modelling, Wendt et al. (2017)

# Stochastic differential equations

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SDEs are Markov processes.

Interpretation of $\mu$ and $\sigma$:

$$\mu_\theta(t, X_t) = \lim_{h \to 0} \frac{1}{h} \mathbf{E}\left[X_{t+h} - X_t | \mathcal{F}_t\right] \tag{3}$$

$$\sigma_\theta(t, X_t)\sigma_\theta(t, X_t)^T = \lim_{h \to 0} \frac{1}{h} \mathbf{Var}\left[X_{t+h} - X_t | \mathcal{F}_t\right] \tag{4}$$

# Bayesian inference

Posterior distribution is given by

$$p(\theta|\vec{X}) \propto p(\vec{X}|\theta)p(\theta) \tag{5}$$

Problem is that $p(\vec{X}|\theta)$ is rarely given in closed form.

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The transition kernel $p(\vec{X}|\theta) = \prod_{n=1}^{N} p_\theta(x_{t_n}|x_{t_{n-1}})$ can be approximated using

- PDE methods (Fokker-Planck)
- Monte Carlo
  - Pedersen sampler (1995)
  - Bridge sampler (2002)
  - Lindström sampler (2012)
  - Residual sampler (2016)

# Computing the transition density

Combine the *law of total probability* with Bayes formula and the fact that diffusions are *Markov processes*

$$p_\theta(x_T|x_0) = \int p_\theta(x_T, x_s|x_0)\mathrm{d}x_s \quad = \int p_\theta(x_T|x_s)p_\theta(x_s|x_0)\mathrm{d}x_s$$
$$= \mathbf{E}\left[p_\theta(x_T|x_s)|x_0\right].$$

Generate empirical version of $p_\theta(x_s|x_0)$ using Monte Carlo

$$p_\theta^K(x_s|x_0) = \frac{1}{K}\sum_{k=1}^{K}\delta(x_s - x_s^k) \tag{6}$$

resulting in, Pedersen (1995),

$$\hat{p}_\theta(x_T|x_0) = \frac{1}{K}\sum_{k=1}^{K}p_\theta(x_T|x_s^k) \tag{7}$$

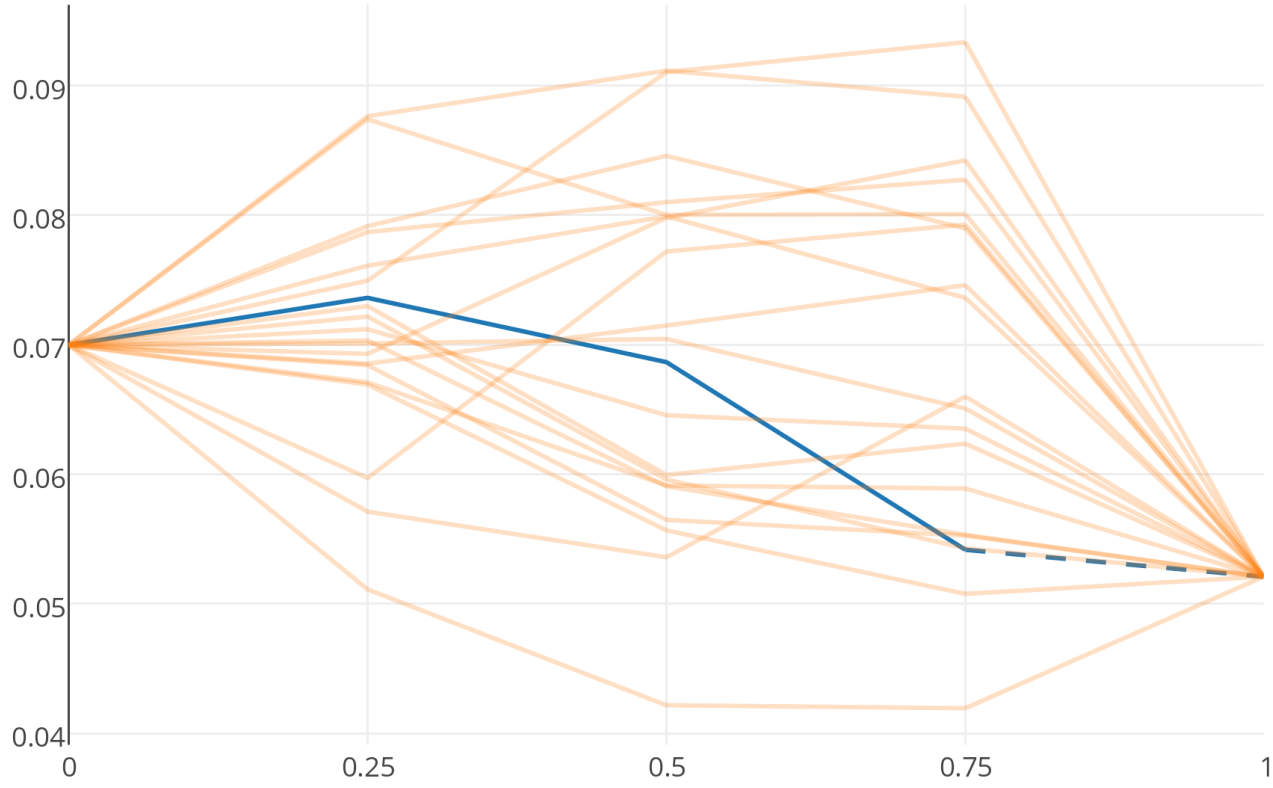


Figure: Example of the simulated maximum likelihood algorithm introduced by Pedersen (1995) using $K = 20$ trajectories on a grid with $R = 4$ equal intervals

# Complexity

In practice, there are two types of errors, discretization and variance

$$\begin{aligned}\epsilon &= \hat{p}_\theta(x_T^\delta|x_0) - p_\theta(x_T|x_0) \\ &= \underbrace{\hat{p}_\theta(x_T^\delta|x_0) - \hat{p}_\theta(x_T|x_0)}_{\textit{Bias}} + \underbrace{\hat{p}_\theta(x_T|x_0) - p_\theta(x_T|x_0)}_{\textit{Random error}}\end{aligned}$$

Simulating $K$ Monte Carlo trajectories, time partitioned in $R$ equal parts (s.t. $\delta = T/R$) gives

- Bias $\mathcal{O}(1/R)$
- Variance $\mathcal{O}(1/K)$
- Mean square error $\mathbf{E}[\epsilon^2]$ is then $\mathcal{O}(1/R^2 + 1/K)$
- Computational complexity for RMSE error of $\mathcal{O}(\epsilon)$ in $\mathcal{O}(\epsilon^{-3})$

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Exact simulation would give a computational complexity $\mathcal{O}(\epsilon^{-2})$

# Multilevel Monte Carlo

Giles (2008) showed organizing computations in a clever way reduces complexity. The MLMC method reduces the cost for obtaining an RMSE of $\mathcal{O}(\epsilon)$ to $\mathcal{O}(\epsilon^{-2}(\log \epsilon)^2)$.

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The idea is basically *control variates*, but refined.

# Control variates

- Suppose we want to estimate $\mathbf{E}[X]$, and that we also have $Y$ as well as $\mathbf{E}[Y]$.
- Define

$$\xi = X + \gamma(Y - \mathbf{E}[Y]). \tag{8}$$

  Then $\mathbf{E}[\xi] = \mathbf{E}[X]$.
- $\gamma$ can be chosen such that the variance is minimized

$$\mathbf{Var}[\xi] = \mathbf{Var}[X](1 - \rho^2) \tag{9}$$

# Example - fig tree

My sister can grow figs in her garden - Can I grow figs in my garden?

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My sister can grow figs in her garden - Can I grow figs in my garden?
Estimate the average temperature in my garden, $\mathbf{E}[X]$, given the measurements last year $X$ and the temperatures $Y$ and average temperature where she lives $\mathbf{E}[Y]$.

# Multi Level Monte Carlo

- Compute the expectation as a sequence of correction terms at levels $l = 0, \ldots, L$
- Denote the approximation at level $l$ by $P_l$.
- Then $\xi$ is defined as

$$\xi = P_0 + \sum_{l=1}^{L} (P_l - P_{l-1}) \tag{10}$$

where $\mathbf{E}[\xi] = \mathbf{E}[P_L]$.

# Multiple levels

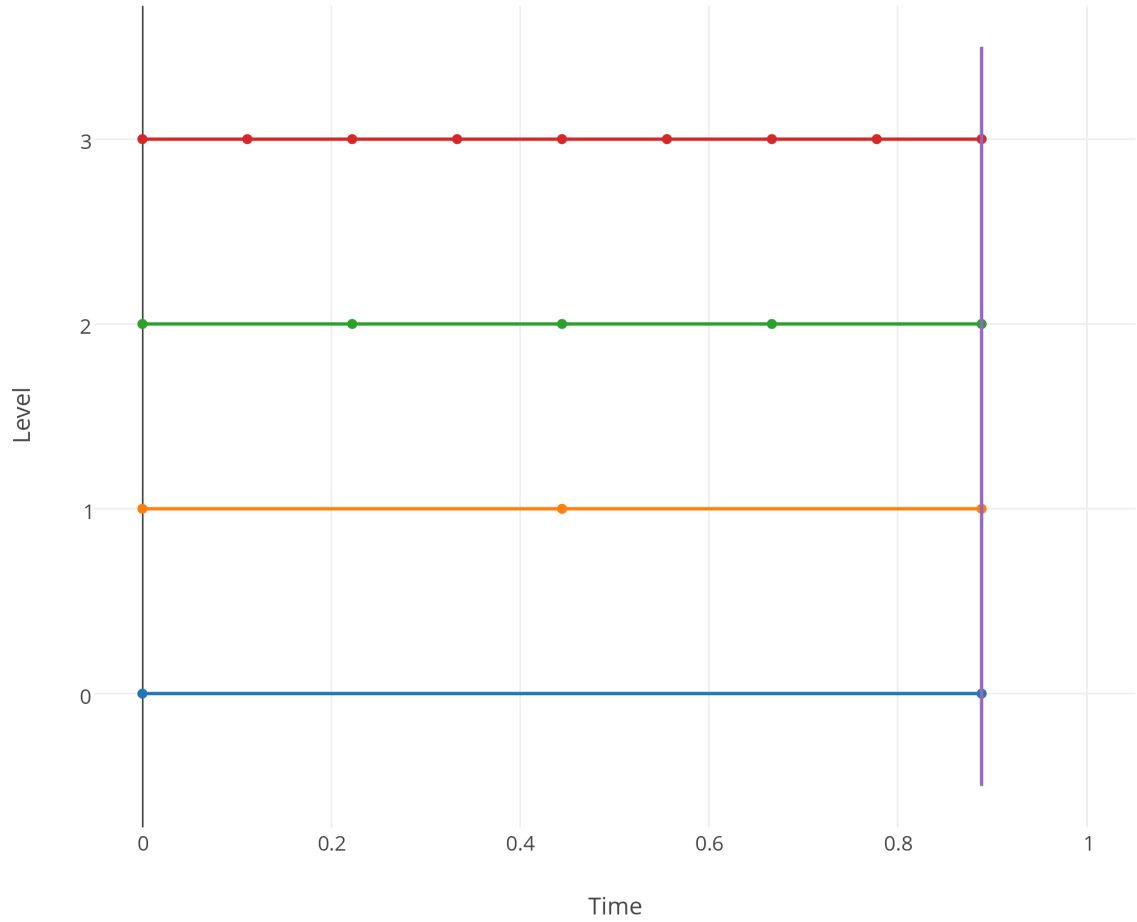


Figure: Multilevel grid example for the Pedersen algorithm. Here we use $L = 3$ levels with $M = 2$. $s$ is indicated by the vertical line.

# MLMC 2

- A naive implementation *increases* the variance
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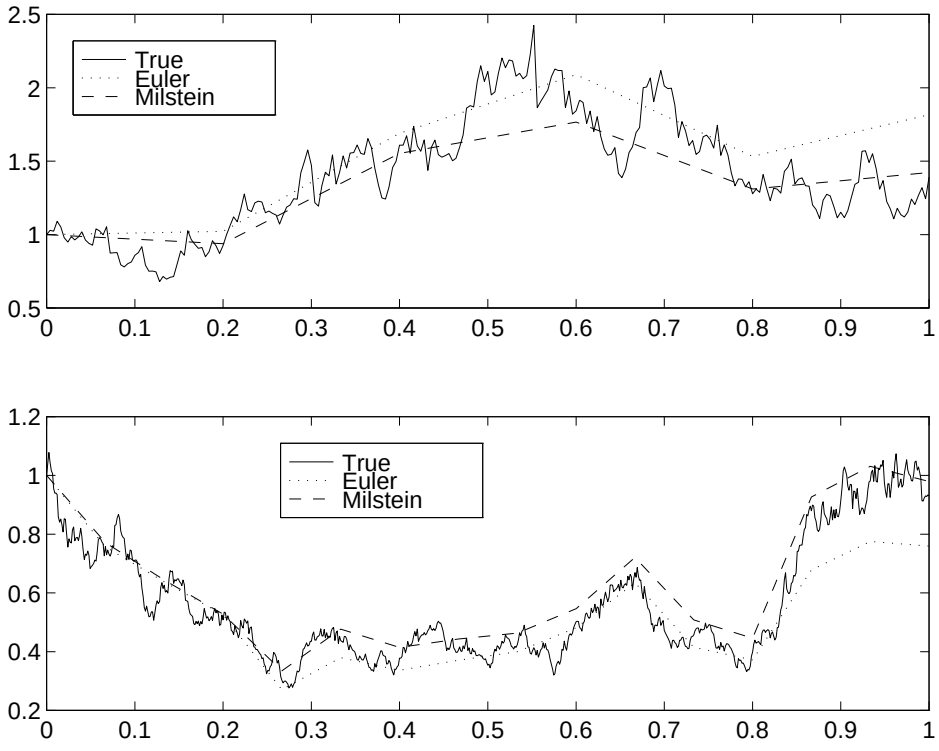
- However, the variance of each $P_l - P_{l-1}$ is very small, and decays with the step size if the use the *same* random elements. Draw a figure to see this!
- This follows from the strong convergence of the discretization scheme.
- Both weak and strong rates of converges are important!

Use Pederson idea in MLMC framework, see (Lindström & Åkerlindh, 2016)**?**.

# Why does it work?




Note the difference between different Brownian motions vs. different schemes/time steps.

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# Comparison

Keep the bias fixed, while comparing variances

## Standard Pedersen

- Cost: $K_p M^L$
- Variance: $\sigma^2 / K_p$

## Multilevel Pedersen

- Cost: $\sum_{l=0}^{L} K_l M^l = K(L+1)$
- Variance: $(L+1)\sigma^2 / K$

Equal cost gives $K = K_p M^L / (L+1)$

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Equal cost gives $K = K_p M^L/(L+1)$. That leads to

$$\frac{\mathbf{Var}[MLMC]}{\mathbf{Var}[MC]} = \frac{(L+1)^2}{M^L} \tag{11}$$

We are currently working on an improved version that will make the MLMC estimator even more efficient.

# Comparison

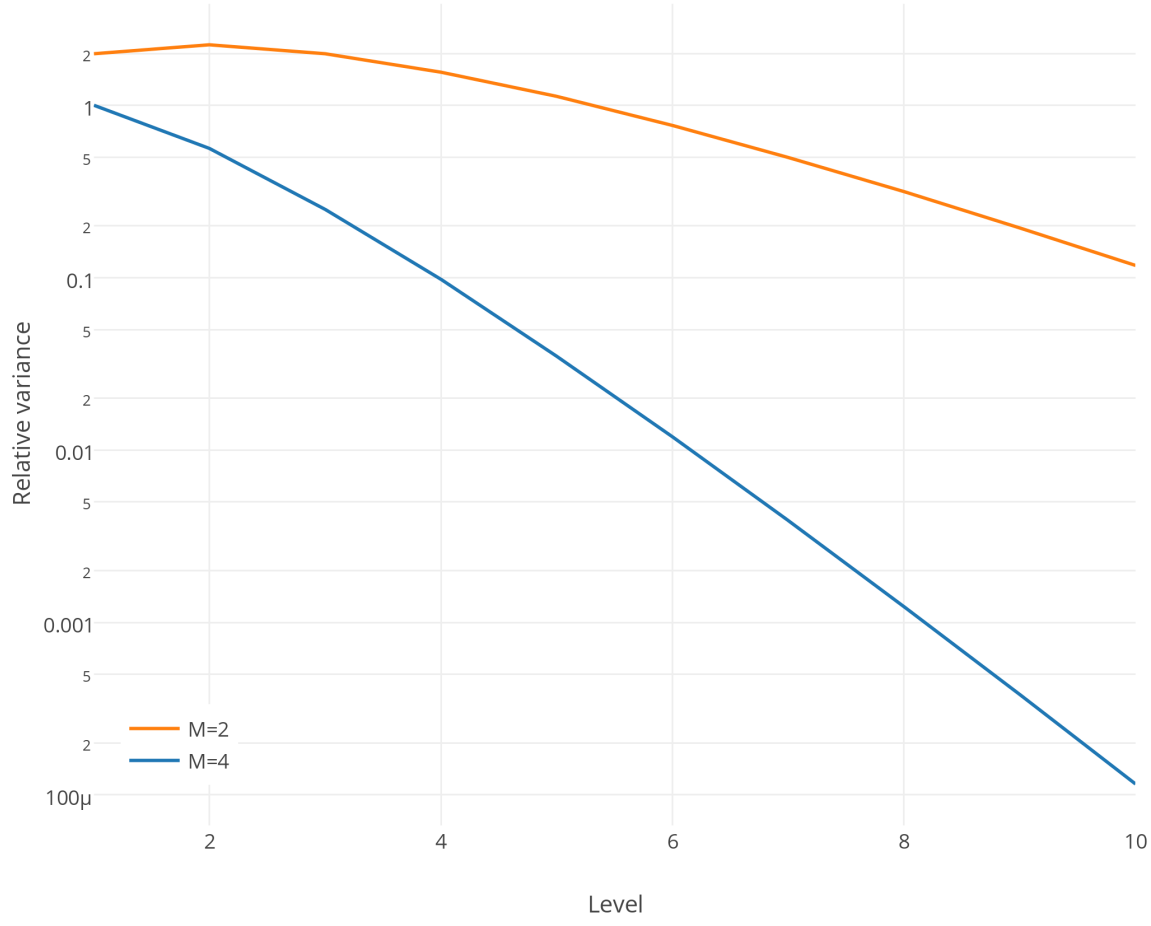


Figure: Theoretical relative variance of standard and multilevel Pedersen with identical compuational cost and bias.

# Simulation study

We consider complete (bivariate) observations from the Heston (1993) model,

$$\mathrm{d}S_t = 0.05 S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^{(S)} \tag{12a}$$

$$\mathrm{d}V_t = 2\left(0.04 - V_t\right)\mathrm{d}t + 0.25\sqrt{V_t}\mathrm{d}W_t^{(V)} \tag{12b}$$

with $\mathrm{d}W_t^{(S)}\mathrm{d}W_t^{(V)} = -0.5\mathrm{d}t$

# Simulation study

- 1000 observations
- Standard Pedersen using $K_p = 2^8 = 256$
- Multilevel Pedersen using $M = 4$ and $L = 3$
- Ground truth, standard Pedersen using $K_{\text{true}} = 10^6$
- Compute

$$\text{VR} = \frac{\sum_{n=1}^{N} \left(\hat{p}_{\text{ML}}(x_n|x_{n-1}) - \hat{p}_{\text{True}}(x_n|x_{n-1})\right)^2}{\sum_{n=1}^{N} \left(\hat{p}_{\text{Pedersen}}(x_n|x_{n-1}) - \hat{p}_{\text{True}}(x_n|x_{n-1})\right)^2} \tag{13}$$

# Simulation study

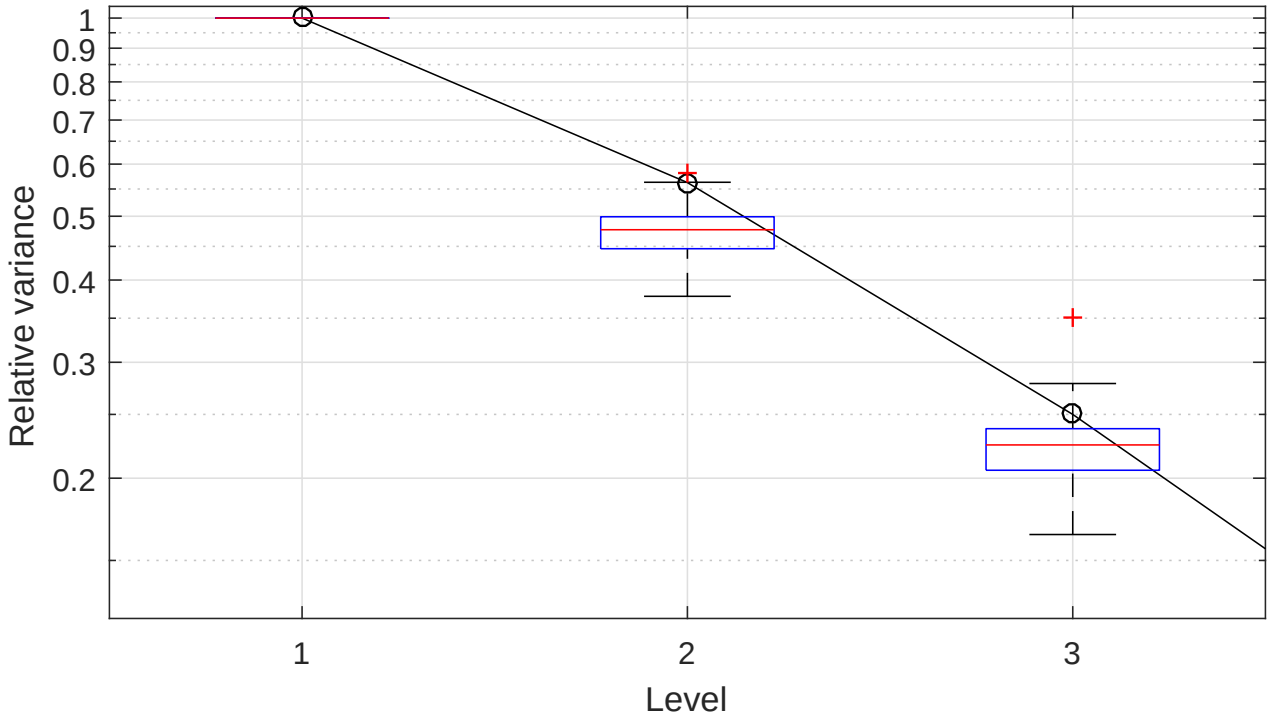


Figure: Relative variance between multilevel and standard Pedersen. Bootstrapped estimates (boxplot) compared to theoretical value (solid line).

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- We could also use the Multi-Level Monte Carlo estimates within an adaptive PMMH algorithm

# Cox-Ingersoll-Ross model

Standard interest rate model until recent years (neg rates)

$$\mathrm{d}r(t) = \kappa\left(\xi - r(t)\right)\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}W(t). \qquad (14)$$

Compare PMMH results

- Exact likelihood (best mixing)
- MC - Pedersen
- MC - Bridge sampler (Durham-Gallant, 2002)
- MLMC - Pedersen

Adaptive MCMC, 2000 iterations as burnin, 2000 additional iterations.

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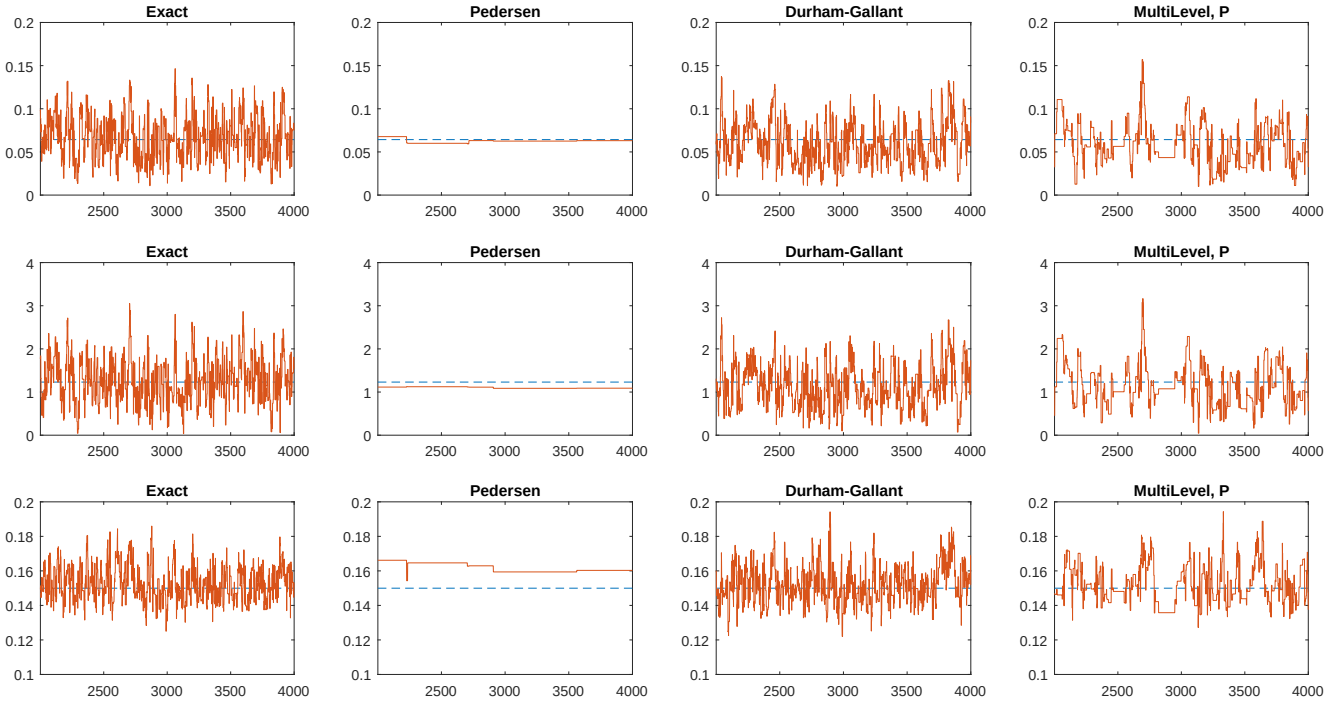
**Prior:** All parameters positive and Feller condition holds.

# Path plots for the parameters




It is known that noisy estimates of the log-likelihood function degrades the mixing of the simulated parameters.

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