LUNDS TEKNISKA HÖGSKOLA **MATEMATIK** **Helsingborg**

LÖSNINGAR **ANALYS 2 FMAA50** **2025-08-26**

1. a) $y' + 2xy = x$. Integrerande faktor är e^{x^2}.

Vi får $e^{x^2} \cdot y' + e^{x^2} 2xy = x \cdot e^{x^2} \Leftrightarrow \dfrac{d}{dx}(e^{x^2} \cdot y) = x \cdot e^{x^2} \Leftrightarrow e^{x^2} \cdot y = \int x \cdot e^{x^2} dx$

Integration ger

$e^{x^2} \cdot y = \dfrac{1}{2} \cdot e^{x^2} + C \Leftrightarrow y = \dfrac{1}{2} + C \cdot e^{-x^2}$.

Svar: $y = \dfrac{1}{2} + C \cdot e^{-x^2}$.

b) $y' = \sin^2 x \Leftrightarrow y' = \dfrac{1}{2}(1 - \cos 2x)$ Integration ger

$$y = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

c) $\dfrac{y'}{x} = -\dfrac{1}{y}$, $y(1) = 2$. Separera variablerna

$y \cdot y' = -x \Leftrightarrow \int y\,dy = \int -x\,dx \Leftrightarrow \dfrac{y^2}{2} = -\dfrac{x^2}{2} + C \Leftrightarrow$

$\Leftrightarrow y^2 = -x^2 + D$. $y(1) = 2$ ger $\Leftrightarrow 2^2 = -1^2 + D \Leftrightarrow D = 5$.

Svar: $y = \sqrt{5 - x^2}$.

2. $y'' - 4y' + 13y = 13x^2 + 18x + 33$ (1) , $y(0) = 3$, $y'(0) = 11$. $y = y_h + y_p$

y_h : $r^2 - 4r + 13 = 0 \Leftrightarrow r = 2 \pm 3i$. Vi får $y_h = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$.

Ansats : $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$, $y_p'' = 2A$

Detta insatt i (1) ger $2A - 4(2Ax + B) + 13(Ax^2 + Bx + C) = 13x^2 + 18x + 33$

$$\begin{cases} 2A - 4B + 13C = 33 \\ -8A + 13B = 18 \\ 13A = 13 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 2 \\ C = 3 \end{cases} \qquad y_p = x^2 + 2x + 3.$$

Den allmänna lösningen är $y = y_h + y_p = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + x^2 + 2x + 3$.

$y' = 2e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + e^{2x}(-3C_1 \sin 3x + 3C_2 \cos 3x) + 2x + 2$.

Villkor $y(0) = 3$, $y'(0) = 11$ ger $\begin{cases} C_1 + 3 = 3 \\ 2C_1 + 3C_2 + 2 = 11 \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ C_2 = 3 \end{cases}$

Svar: $y = y_h + y_p = 3e^{2x} \cdot \sin 3x + x^2 + 2x + 3$.

3. a) $\lim_{x\to 0} \dfrac{\sin x + \arctan x}{x\cdot(1-\cos 2x)} =$

$$= \left| \begin{array}{l} \sin x = x - \dfrac{x^3}{3!} + x^5 B_1(x), \quad \arctan x = x - \dfrac{x^3}{3} + x^5 B_2(x), \\ \cos t = 1 - \dfrac{t^2}{2} + t^4 B_3(t),\ \ t = 2x \ \text{ ger } \ \cos 2x = 1 - \dfrac{(2x)^2}{2} + (2x)^4 B_3(2x) = \\ = 1 - 2x^2 + x^4 B_4(x) \\ \text{där } B_k(x) \text{ begränsad i omgivning av 0.} \end{array} \right| =$$

$$= \lim_{x\to 0} \frac{x - \dfrac{x^3}{6} + x^5 B_1(x) - \left(x - \dfrac{x^3}{3} + x^5 B_2(x) \right)}{x\cdot(1-(1-2x^2+x^4B_4(x))} = \lim_{x\to 0} \frac{\dfrac{x^3}{6} + x^5 B_5(x)}{2x^3 + x^5 B_4(x)} =$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{6} + x^2 B_5(x)}{2 + x^2 B_4(x)} = \frac{\dfrac{1}{6}+0}{2+0} = \frac{1}{12}.$$

b) $f(x) = \sqrt{1+3x^2}$, $f(1) = 2.$ $f'(x) = 3x\left(1+3x^2\right)^{-\frac{1}{2}}$, $f'(1) = \dfrac{3}{2}.$

$$f''(x) = 3\left(1+3x^2\right)^{-\frac{1}{2}} - 3x\cdot\frac{1}{2}\left(1+3x^2\right)^{-\frac{3}{2}}\cdot 6x = 3\left(1+3x^2\right)^{-\frac{1}{2}} - 9x^2\cdot\left(1+3x^2\right)^{-\frac{3}{2}},\ f''(1) = \frac{3}{8}.$$

Taylorpolynomet av ordning 2 i punkten 1 blir:

$$p_2(x) = 2 + \frac{3}{2}\cdot(x-1) + \frac{3}{16}(x-1)^2.$$

4. a) $\displaystyle\int_2^3 \frac{x+1}{x^2-x}\,dx$ $\qquad \dfrac{x+1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$ $\qquad$ Vi får $A=-1$ och $B=2$.

$$\int_2^3 \frac{2}{x-1} - \frac{1}{x}\,dx = \left[2\ln|x-1| - \ln|x|\right]_2^3 = 2\ln 2 - \ln 3 - (2\ln 1 - \ln 2) = 3\ln 2 - \ln 3 = \ln\frac{8}{3}.$$

b) $y(x) = \ln(x^2+1) - \displaystyle\int_0^x \frac{t\cdot y(t)}{t^2+1}\,dt$, $x \ge 0.$

Derivera ekvationen med avseende på x och vi får

$$y'(x) = \frac{2x}{x^2+1} - \frac{x}{x^2+1}y(x) \quad , \quad y(0) = 0.$$

Vi löser differentialekvationen $\quad y'(x) + \dfrac{x}{x^2+1}y(x) = \dfrac{2x}{x^2+1}$

Integrerande faktor är $e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}.$

Vi får $y'(x)\sqrt{x^2+1} + \sqrt{x^2+1}\,\dfrac{x}{x^2+1}\,y(x) = \dfrac{2x}{x^2+1}\sqrt{x^2+1}$

$$\Leftrightarrow \frac{d}{dx}\left(y \cdot \sqrt{x^2+1}\right) = 2x \cdot (x^2+1)^{-\frac{1}{2}} \quad \Leftrightarrow y \cdot \sqrt{x^2+1} = 2\sqrt{x^2+1} + C \Leftrightarrow$$

$$\Leftrightarrow y(x) = 2 + \frac{C}{\sqrt{x^2+1}}. \quad y(0) = 0 \text{ ger } \Leftrightarrow 0 = 2 + C \Leftrightarrow C = -2.$$

Svar: $y(x) = 2 - \dfrac{2}{\sqrt{x^2+1}}.$

5. a) $\displaystyle\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx = \left| \begin{array}{l} t = \sqrt{x+1}\,,\ x = t^2 - 1\,,\ dx = 2t\,dt\,, \\ x = 0 \leftrightarrow t = 1 \text{ och } x = 3 \leftrightarrow t = 2 \end{array} \right| = \int_1^2 \frac{t^2+1}{t} \cdot 2t\,dt = 2\left[\frac{t^3}{3} + t\right]_1^2 = \frac{20}{3}.$

b) $\displaystyle V = \int_1^T \pi\left(2x \cdot e^{-x}\right)^2 dx = \pi\int_1^T 4x^2 \cdot e^{-2x}dx = \pi\left[4x^2 \cdot \left(-\frac{1}{2}\right)e^{-2x}\right]_1^T - \pi\int_1^T 8x \cdot \left(-\frac{1}{2}\right)e^{-2x}dx =$

$$= \pi\left[-2x^2 \cdot e^{-2x}\right]_1^T + \pi\int_1^T 4x \cdot e^{-2x}dx = = \left[-2x^2 \cdot e^{-2x}\right]_1^T +$$

$$+\left[-\frac{1}{2}e^{-2x} \cdot 4x\right]_1^T - \int_1^T -\frac{1}{2}e^{-2x} \cdot 4dx =$$

$$= \left[-(2x^2+2x) \cdot e^{-2x}\right]_1^T + \int_1^1 2e^{-2x}dx = \left[-(2x^2+2x) \cdot e^{-2x}\right]_1^T + \left[-e^{-2x}\right]_1^T =$$

$$= \left[-(2x^2+2x+1) \cdot e^{-2x}\right]_1^T = \pi\left(-(2T^2+2T+1) \cdot e^{-2T} - \left(-5e^{-2}\right)\right) \to \frac{5\pi}{e^2} \quad , \quad T \to \infty.$$

Svar: $\displaystyle V = \int_1^\infty \pi\left(2x \cdot e^{-x}\right)^2 dx = \frac{5\pi}{e^2}$

6. Ekvationen är $v' = -10 - 2{,}5 \cdot 10^{-4}v^2 \;\Leftrightarrow\; v' = -10\left(1 + 25 \cdot 10^{-6}v^2\right) \;\Leftrightarrow\; \dfrac{v'}{1 + 25 \cdot 10^{-6}v^2} = -10\,.$

$\displaystyle\int \frac{dv}{1 + 25 \cdot 10^{-6}v^2} = -10\int dt \Leftrightarrow 200\arctan\frac{v}{200} = -10t + C.$ Villkoret $v(0) = 200$ ger $C = 50\pi$.

Alltså är $20\arctan\dfrac{v}{200} = 5\pi - t$. Projektilen stiger tills $v = 0$ dvs $0 = 5\pi - t \Leftrightarrow t = 5\pi$.

SLUT!