A crash course in Finite Elements
PhD Seminars VT22

Jaime R. Manríquez

LTH, Faculty of Engineering
Lund University

March, 2022

Outline

A standard equation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2\}$. We are looking for $u \in H^1(\Omega)$ such that

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= u_D \text{ on } \Gamma_D, \\
\frac{\partial u}{\partial \boldsymbol{\nu}} &= q \quad \text{on } \Gamma_N,
\end{aligned}
$$

where $\partial\Omega := \Gamma_D \cup \Gamma_N$, $f \in L^2(\Omega)$, $u_D \in H^{1/2}(\Gamma_D)$ and $q \in H^{-1/2}(\Gamma_N)$.

A weaker version

Let $v \in H^1_D(\Omega) \subset H^1(\Omega)$. We have that

$$\int_\Omega fv = \int_\Omega (-\Delta u)v$$

$$= \int_\Omega \nabla u \cdot \nabla v - \cancelto{0}{\int_{\Gamma_D} \frac{\partial u}{\partial \boldsymbol{\nu}} v} - \int_{\Gamma_N} \frac{\partial u}{\partial \boldsymbol{\nu}} v.$$

Using the boundary conditions, we arrive at

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_N} qv$$

or, equivalently,

$$a(u,v) = F(v) + Q(v) \qquad \forall v \in H^1_D(\Omega).$$

Our problem

Take $u_D \equiv 0$ and $\Gamma_N = \emptyset$. Now we are looking for $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$$

or, equivalently,

$$a(u, v) = F(v) \qquad \forall v \in H_0^1(\Omega) \tag{WF}$$

Lax-Milgram lemma

Let V be a real Hilbert space and $a : V \times V \to \mathbb{R}$ a continuous bilinear form that is V-coercive with constant α. Then, for all $F \in V'$ there exists a unique $u \in V$ such that

$$a(u, v) = F(v) \qquad \forall v \in V.$$

Furthermore, $\|u\| \leq \alpha^{-1} \|F\|$.

The Galerkin method

To approximate $u \in V$, we consider a sequence of finite dimensional subspaces $V_1 \subset V_2 \subset \ldots$ such that $\overline{\bigcup_{n=1}^{N} V_n} = V$ and find the solution $u_n \in V_n$ of

$$a(u_n, v_h) = F(v_h) \qquad \forall v_h \in V_n \tag{DF}$$

such that this sequence satisfies $\lim_{n\to\infty} \|u - u_n\| = 0$.
If now we implement the systematic approach of dividing our subdomain Ω in a family of *discretizations* $\{\Omega_h\}_{h>0}$ and let our subspaces be spaces of polynomials $\mathbb{P}^k$, we have what we call **a finite element method**.

The Galerkin method (II)

Since V_h is finite dimensional, we can take a basis $\{\phi_j\}_{j=1}^N$ and solve the Galerkin system

$$\mathbb{A}\boldsymbol{\beta} = \mathbf{F},$$

where $\mathbb{A}_{ij} = a(\phi_j, \phi_i)$, $\mathbf{F}_i = F(\phi_i)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)^t$ is such that $u_h = \sum_{j=1}^N \beta_j \phi_j$.
In some contexts, they call $\mathbb{A}$ *the stiffness matrix*, $\boldsymbol{F}$ *the load vector* and $\boldsymbol{\beta}$ *the degrees of freedom*.

The Galerkin method (III)

For example, take $\Omega =]0,1[$ and a partition $x_i = ih$ with $0 < h < 1$. Let V_h be the space of continuous functions that are polynomials of degree 1 when restricted to $[x_i, x_{i+1}]$.
Consider a basis $\{\phi_j\}_{j=1}^N$ such that $\phi_j(x_i) = \delta_{ij}$. We call these functions *hat functions* because of their shape and notice that ϕ_j is supported only on $[x_{j-1}, x_{j+1}]$.

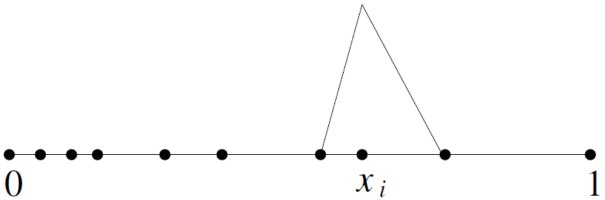

Figure: Image from [SCB08]

The Galerkin method (IV)

We have that

$$a(\phi_j, \phi_i) = \begin{cases} 2/h & \text{if } |i-j| > 1 \\ -1/h & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

and so we end up solving the system

$$\frac{1}{h} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N. \end{pmatrix}$$

A more thorough explanation of implementation can be found in [ACF99].

A Finite Element

Definition (Classical, [Cia02, SCB08])

A finite element is a triple $(K, \mathcal{P}, \mathcal{N})$ where

- $K \subset \mathbb{R}^n$ is a compact set with nonempty interior and piece-wise smooth boundary (the *element domain*),
- $\mathcal{P}$ is a finite dimensional space of functions on K (the space of *shape functions*) and
- $\mathcal{N} = \{\xi_1, \ldots, \xi_N\}$ is a basis of $\mathcal{P}'$ (the *degrees of freedom*.)

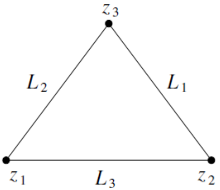

Fig. 3.1. linear Lagrange triangle

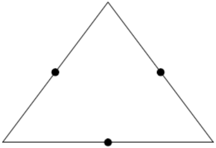

Fig. 3.2. Crouzeix-Raviart nonconforming linear triangle

A Finite Element (II)

Definition (Unisolvency)

A set $\mathcal{N} = \{\xi_1, \ldots, \xi_d\} \subset \mathcal{P}'$ is called *unisolvent* if $\xi_i(v) = 0$ for $i = 1, \ldots, d$ implies $v \equiv 0$. If $d = \dim \mathcal{P}$, then $\mathcal{N}$ is a basis of $\mathcal{P}'$.

Definition (Nodal basis)

The *nodal* basis is the basis $\{\phi_1, \ldots, \phi_N\}$ of $\mathcal{P}$ dual to $\mathcal{N}$, that is, $\xi_j(\phi_i) = \delta_{ij}$.

Definition (Local interpolant)

Given $(K, \mathcal{P}, \mathcal{N})$, we define the *local interpolant* by

$$\mathcal{I}_K := \sum_{j=1}^{N} \xi_j(v)\phi_j.$$

Error in H^1-norm

Since u is a solution of the original weak problem and $V_h \subset V$, it follows that $a(u, v_h) = F(v_h)$ for all $v_h \in V_h$ and so we conclude:

Theorem (Galerkin orthogonality)

Let u and u_h be the solutions of (WF) *and* (DF), *respectively. Then*

$$a(u - u_h, v_h) = 0$$

for all $v_h \in V_h$.

Theorem (Céa's lemma)

We have that

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v\|_V .$$

Error in H^1-norm (II)

Theorem (Bramble-Hilbert lemma [Bra07])

Let $t \geq 2$ and suppose Ω_h is a regular discretization of Ω. Then there exists a constant C independent of m such that

$$\|u - \mathcal{I}u\|_{m,h} \leq Ch^{t-m} |u|_{t,\Omega}$$

for all $u \in H^t(\Omega)$ and $0 \leq m \leq t$, where $\mathcal{I}$ denotes interpolation by a piecewise polynomial of degree $t-1$.

Error in H^1-norm (III)

Consider the $\mathbb{P}^1$ Lagrange element. We can use the Bramble-Hilbert lemma to deduce that

$$\|u - \mathcal{I}u\|_{1,\Omega} \leq Ch\,|u|_{2,\Omega}\,.$$

Then, since the interpolant projects onto the space V_h we have $\mathcal{I}u \in V_h$ and so, from Céa's lemma,

$$\|u - u_h\|_{1,\Omega} \leq \frac{B}{\alpha}\,\|u - \mathcal{I}u\|_{1,\Omega} \lesssim h\,\|u\|_{2,\Omega}\,.$$

And finally, if we possess some (say, elliptic) regularity with $f \in L^2(\Omega)$, we can arrive at

$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h).$$

Error in L^2-norm

Theorem (Aubin-Nitsche [Bra07])

Let H be a Hilbert space w.r.t. norm $\|\cdot\|$ and $V \subset H$ be a subspace which is also Hilbert under another norm $|\cdot|$. In addition, let $V \hookrightarrow H$ be continuous.
Then, the finite element solution of (DF) satisfies

$$\|u - u_h\| \leq C\,|u - u_h| \sup_{g \in H} \left\{ \frac{1}{\|g\|} \inf_{v \in V_h} |\varphi_g - v| \right\},$$

where for every $g \in H$, $\varphi_g \in V$ denotes the corresponding unique weak solution of

$$a(w, \phi_g) = (g, w) \qquad \forall w \in V.$$

Error in L^2-norm (II)

In our setting, we have that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and so

$$\begin{aligned}\|u-u_h\|_{0,\Omega} &\leq C\,\|u-u_h\|_{1,\Omega} \sup_{g\in L^2(\Omega)} \left\{ \|g\|_{0,\Omega}^{-1} \inf_{v\in V_h} \|\varphi_g - v\|_{1,\Omega} \right\} \\ &\lesssim h\,\|u\|_{2,\Omega} \sup_{g\in L^2(\Omega)} \left\{ \|g\|_{0,\Omega}^{-1}\, h\,\|g\|_{0,\Omega} \right\} \\ &\lesssim h^2\,\|u\|\end{aligned}$$

and so

$$\|u-u_h\|_{0,\Omega} = \mathcal{O}(h^2).$$

- J. Alberty, C. Carstensen, and S. A. Funken, *Remarks around 50 lines of Matlab: short finite element implementation*, Numer. Algorithms **20** (1999), no. 2–3, 117–137. MR 1 709 562
- Dietrich Braess, *Finite elements*, Cambridge University Press, 2007.
- Philippe G. Ciarlet, *The finite element method for elliptic problems*, 2nd ed., Classics in applied mathematics 40, Society for Industrial and Applied Mathematics, 2002.
- Alexandre Ern and Jean-Luc Guermond, *Finite elements i*, Springer International Publishing, 2021.
- L. Ridgway Scott Susanne C. Brenner, *The mathematical theory of finite element methods*, 3 ed., Texts in Applied Mathematics, Springer-Verlag New York, 2008.