Exercises: Initial value problems

Numerical Analysis, FMNF10, 2018

1. Consider the initial value problem

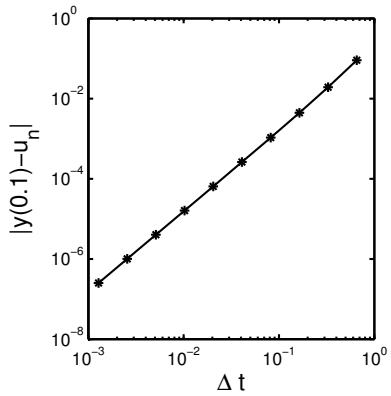
$$y'(t) = -y(t)^2 + \cos(t), \quad t > 0,$$

with the initial value $y(0) = 2$.

a) Approximate $y(0.1)$ by using the explicit Euler method with the step size $\Delta t = 0.05$.

b) Explain what is meant by the statement: "The explicit Euler method has a convergence order of $p = 1$".

c) Your friend has implemented a different method and solved the problem for various values of Δt. The resulting global errors $|y(0.1) - u_n|$ are given in the graph below, where u_n denotes the approximation of $y(0.1)$. Estimate the convergence order p of your friend's method.



2. Rewrite the second order differential equation

$$v''(t) = v(t)^2 + v'(t), \quad v(0) = 1, \quad v'(0) = 0,$$

as a system of first order equations (standard form). Do not forget the initial value.

3. Consider the MATLAB program

```
lambda = -1000;
N = 10;
u(1) = 1;
dt = 1/N;
for n = 1:N
    u(n+1) = u(n) + dt*(lambda*u(n));
end
```

a) Which numerical method is implemented in the code, and which problem does it solve?

b) If we would run the code it would give a very disappointing result. Explain why this is the case.

c) Modify the code (without changing the problem being solved) so that the numerical approximation is improved.

4. The stability of a time stepping method can be checked by applying it to the test equation

$$y'(t) = \lambda y(t), \quad y(0) = \alpha.$$

When this problem is approximated one obtains a recursion

$$u^{n+1} = R(\lambda \Delta t) u^n,$$

where R is a polynomial if the method is explicit and a rational function, i.e., the quotient of two polynomials, if the method is implicit. The stability region is then defined as the set

$$S = \{\lambda \Delta t \in \mathbb{C} : |R(\lambda \Delta t)| \leq 1\}.$$

Find and plot the stability region in the complex plane for the trapezoidal rule, where

$$u^{n+1} = u^n + \frac{1}{2}\Delta t\big(f(t_n, u^n) + f(t_{n+1}, u^{n+1})\big).$$

5. The classical Runge–Kutta schemes is given by

$$\begin{aligned}
k_1 &= f(t_n, u_n), \\
k_2 &= f\big(t_n + \tfrac{1}{2}\Delta t, u_n + \tfrac{1}{2}\Delta t\, k_1\big), \\
k_3 &= f\big(t_n + \tfrac{1}{2}\Delta t, u_n + \tfrac{1}{2}\Delta t\, k_2\big), \\
k_4 &= f\big(t_n + \Delta t, u_n + \Delta t\, k_3\big), \\
u^{n+1} &= u^n + \frac{1}{6}\Delta t\,\big(k_1 + 2k_2 + 2k_3 + k_4\big).
\end{aligned}$$

a) Write a Matlab function

```
function un_plus_1 = RK(dt,tn,un)
```

that evaluates one step of this method.

b) Write a Matlab function that computes $u_1, \ldots, u_N$, by calling the function `RK`.

c) Use your implementation to approximate a solution of the initial value problem

$$y_1'(t) = y_2(t)$$
$$y_2'(t) = -y_1(t) + y_2(t)\big(1 - y_1(t)^2\big).$$

Use the step size $\Delta t = 0.05$ over the time interval $[0, 100]$ and set, e.g., $y(0) = (0,\ 1)^{\mathrm{T}}$.

d) Plot y_2 as a function of y_1. By looking at the plot, can you say something about the behavior of y as time tends to infinity?