Exercises: Boundary value problems

Numerical Analysis, FMNF10, 2018

Numerical differentiation
1. Prove that the symmetric differences is a second order approximation of the second derivative, i.e.,
\[ \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2} - f''(x) = \mathcal{O}(\Delta x^2), \]
for a sufficiently smooth function \( f \).

2. We shall approximate the derivate \( f' \) of a function \( f \) by the formula
\[ f'(x) \approx \frac{4f(x + \Delta x) - 3f(x) - f(x - 2\Delta x)}{6\Delta x}. \]
Determine the order of the approximation formula.

Discretizations of boundary value problems
3. We want to solve the linear boundary value problem
\[ y''(x) + (1 + x^2) y'(x) + y(x) = 0, \]
with boundary values \( y(0) = 1 \) and \( y(1) = 2 \). Introduce a suitable notation and discretize the problem by a standard second order method; compare with Problem 1. Construct the linear system in matrix-vector form that has to be solved. Make sure to include the boundary conditions and state the matrix dimensions, vector lengths, spatial grid and compute the step size \( \Delta x \).

4. Not all boundary value problem are linear. As an example consider the equation
\[ y''(x) + y(x)(y(x) + 1) = 0, \]
with boundary values \( y(0) = 0 \) and \( y(2) = 2 \). Why is it not linear? Luckily, we can still discretize nonlinear boundary problems. The difference is that we no longer obtain a linear system \( Au = b \). Instead we obtain a system of nonlinear equations \( F(u) = 0 \), and a solution can be approximated by Newton’s method.

a) Introduce a suitable notation and discretize the nonlinear boundary value problem. Construct the system of nonlinear equations \( F(u) = 0 \) that have to be solved. Be sure to state the number of equations, the grid, the step size \( \Delta x \), etc.

b) Construct the Jacobian matrix \( F'(u) \) and write out the Newton iterations for the problem at hand.

Remark Training on implementing discretizations of boundary value problems is the theme of Project 2.