Exercises: Least squares and numerical integration

Numerical Analysis, FMNF10, 2018

Least squares method

1. We would like to fit a straight line $y(x) = c_0 + c_1x$ to the data below by using the least squares method.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

a) Derive the overdetermined system $Ac \approx b$, which the coefficients $c = (c_0, c_1)^T$ need to fulfill.

b) Approximate a solution to the overdetermined system by the least squares method.

c) Plot the data points together with your straight line. Is the line a good fit to the data?

2. Consider the experimental measurements

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.00</th>
<th>1.00</th>
<th>2.00</th>
<th>3.00</th>
<th>4.00</th>
<th>5.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.03</td>
<td>0.15</td>
<td>0.89</td>
<td>2.79</td>
<td>6.42</td>
<td>12.5</td>
</tr>
</tbody>
</table>

a) Derive the overdetermined system that a polynomial of degree $n$, with $0 \leq n \leq 5$, needs to satisfy in order to fit the data points.

b) Write a code that approximates a solution to the overdetermined system by the least squares method for $n = 0, 1, \ldots, 5$.

c) Plot the data points together with the computed polynomials. Which polynomial seems to capture the general trend in the data?

Numerical integration

3. Consider the following MATLAB code:

```matlab
N = 3;  
a = 0;  \quad b = 1;  
dx = (b-a)/N;  
x = linspace(a+dx/2,b-dx/2,N)';  
fx = cos(x);  
weights = ones(1,N);  
int = dx * weights * fx;  
```
a) Which numerical method is implemented in the code, and what does the quantity \texttt{int} approximate?

b) Modify the code (without changing the problem being solved) so that the numerical approximation is improved.

4 Consider the integral
\[
\int_0^1 \sin(x^2) \, dx.
\]

a) Approximate the integral by employing the midpoint rule and dividing the interval \([0, 1]\) into \(N = 3\) equally sized intervals.

b) How does the error of the midpoint rule decrease as the number of intervals \(N\) tends to infinity?

5 Consider the integral
\[
\int_0^1 x^2(x + 1) \, dx.
\]

a) Approximate the integral by employing Simpson’s rule and dividing the interval \([0, 1]\) into \(N = 2\) equally sized intervals.

b) Why is the approximation error equal to zero when applying Simpson’s rule to the integral above?

6. Consider the approximation of the integral
\[
\int_0^1 \frac{1}{1 + 25x^2} \, dx = 0.274680\ldots
\]

a) Compute numerically the integral above by employing the trapezoidal rule and dividing the interval \([0, 1]\) into \(N = 4\) equally sized intervals.

b) Write a code for the trapezoidal approximation of the integral for a general \(N\). How large do you need to choose \(N\) in order to obtain an answer with six correct digits?