

## Exercises: Nonlinear equation solving

Numerical Analysis, FMNF10, 2018

### Fixed point iteration

1. Consider the nonlinear equation  $x = \cos(x)$  and the corresponding fixed point iterations, i.e.,

$$x_{n+1} = \cos(x_n), \quad n = 0, 1, 2, \dots$$

- a. Write a code that draws  $y = \cos(x)$  and  $y = x$ . Use the figure to choose a starting value  $x_0$ .
- b. Implement the fixed point iteration in such a way that the code takes the number of iterations, say  $N$ , and the starting value  $x_0$  as inputs, and returns the vector  $(x_1, x_2, \dots, x_N)$  as output.
- c. Run your code for  $N = 50$ . Does  $x_{50}$  seem to be a reasonable approximate solution for the equation  $x = \cos(x)$ ?

2. Next consider the equation

$$x = \sqrt{x+3}, \quad 1 \leq x \leq 3.$$

In order to approximate a solution, we will once more try to use fixed point iterations.

- a. Modify your code from Exercise 1 and perform  $N = 50$  iterations with  $x_0 = 2$  as starting value. Does the fixed point iteration seem to converge?
  - b. Prove that your fixed point iteration is either convergent or divergent (depending on what you have observed in 2a).
3. According to our theory, the error in the fixed point iteration can be bounded as

$$|x_{n+1} - x^*| \leq \frac{m}{1-m} |x_n - x_{n+1}|,$$

where  $x^*$  is the solution of  $x = g(x)$  and  $|g'(x)| \leq m < 1$ .

- a. Use the above error bound in order to modify your code from Exercise 1 in such a way that it takes a tolerance `To1` as input, instead of a fixed number of iterations  $N$ , and iterates until  $|x_n - x^*| \leq \text{To1}$ .
- b. Try out the code on  $x = \sqrt{x+3}$  with  $x_0 = 2$  and `To1` =  $10^{-12}$ . Here, we can, e.g., choose  $m = 1/4$  (explain why!). How many iterations did the code use? What is the advantage with this approach compared to choosing a fixed number of iterations?

### Newton's method

4. You are given the following *incomplete* implementation of Newton's method.

```
x(1) = 1 ;
N = 50 ;
for n = 1:N
    f_x(n) = exp(-x(n)) - x(n) ;
-->    fprim_x(n) =
-->    x(n+1) =
end
```

- Complete the implementation by filling in the blanks (marked with -->).
  - The program ends after  $x(51)$  is computed. What does the quantity  $x(51)$  approximate?
5. Use the above implementation of Newton's method, or write your own, for the equations

$$\cos(x) - x = 0 \quad \text{and} \quad x - \sqrt{x+3} = 0.$$

Choose the same starting values  $x_0$  as in Exercises 1a and 2a, respectively, and  $N = 50$ . Do you see any speed-up when you use Newton's method compared to your fixed point iterations in Exercise 1 and 2?

6. Consider the equation

$$x^3 - a = 0,$$

where  $a$  is a given real number.

- Construct the Newton iteration formula for this problem.
- Let  $a = 1/27$  and find the solution  $x^*$  analytically. Run your Newton code with  $x_0 = 4$  and generate  $N = 10$  iterations, i.e.,  $(x_1, x_2, \dots, x_{10})$ . As you know the exact solution  $x^*$ , you can compute the errors

$$\varepsilon_n = x_n - x^*, \quad n = 0, 1, \dots, 10.$$

Try this and then compute

$$\frac{|\varepsilon_{n+1}|}{|\varepsilon_n|}, \quad \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^2} \quad \text{and} \quad \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^3}, \quad n = 0, 2, \dots, 9.$$

By looking at these results, can you draw any conclusion regarding the convergence order of Newton's method?

- Let  $a = 0$  and redo Exercise 3b. Do you get the same convergence order? If not, explain why!

**7 [Extra in case you have some time over].** In class we have learned that Newton's method converges "if the starting values are close enough" to the root. This can be illustrated for the problem

$$f(x) = \arctan(x) = 0,$$

which has the single root  $x = 0$ .

- a. Construct the Newton iteration formula for this problem.
- b. That formula can be written as a fixed point iteration,  $x_{n+1} = g(x_n)$ . The Newton iteration will converge if  $|x_0| < \alpha$ , where  $\alpha$  is to be computed. It is determined by the condition

$$\alpha = -g(\alpha).$$

Draw a graph of the original problem, and draw what happens in a Newton iteration, in order to interpret this characterization, so that you understand what it means in your graph. Then construct the equation for  $\alpha$ .

- c. Solve the equation for  $\alpha$  using Newton's method. (Note that this means that you will have to construct yet another Newton iteration formula.)