Time Series Analysis

Andreas Jakobsson

Content:

- Linear prediction
- Prediction of ARMA processes
- Prediction of ARMAX processes

Linear prediction

A *linear* prediction is formed as a weighted sum of earlier observations, such that

$$\hat{y}_{t+k|t} = \sum_{\ell=0}^{n} w_\ell y_{t-\ell}$$

For a Gaussian process, the optimal predictor is linear, and is thus the same as the optimal linear predictor. This predictor is determined such that

$$C\Big\{y_{N+k} - \hat{y}_{N+k|N}, y_t\Big\} = C\Big\{y_{N+k} - \sum_{\ell=0}^{n} w_\ell y_{N-\ell}, y_t\Big\} = 0$$

That is, the optimal predictor is such that the resulting prediction error is uncorrelated with the earlier observed measurements. This is but a reformulation of the orthogonality principle.

Linear prediction

The optimal (linear) k-step prediction error is

$$\begin{aligned}\mathcal{C} &= E\left\{\left[y_{N+k} - \hat{y}_{N+k|N}\right]^2\right\} = V\left\{y_{N+k} - \hat{y}_{N+k|N}\right\} \\ &= C\left\{y_{N+k} - \hat{y}_{N+k|N}, y_{N+k}\right\} \\ &= V\left\{y_{N+k}\right\} - \sum_{\ell=0}^{n} w_\ell C\left\{y_{N-\ell}, y_{N+k}\right\}\end{aligned}$$

as $y_{N+k} - \hat{y}_{N+k|N}$ is orthogonal to $\hat{y}_{N+k|N}$.

Linear prediction

Example: Consider the MA process $y_t = e_t + c_1 e_{t-1}$, and assume that y_1 and y_2 have been observed. The optimal linear prediction of y_3 can then be found by noting that

$$C\{y_3 - w_1 y_1 - w_2 y_2, y_k\} = 0$$

for $k = 1$ and $k = 2$. Thus,

$$\begin{aligned}r_y(2) - w_1 r_y(0) + w_2 r_y(1) = 0 \\ r_y(1) - w_1 r_y(1) + w_2 r_y(0) = 0\end{aligned}$$

implying that

$$r_y(0) = (1 + c_1^2)\sigma_e^2 \quad r_y(1) = c_1 \sigma_e^2 \quad r_y(2) = 0$$

and thus

$$w_1 = -\frac{\rho_y^2(1)}{1 - \rho_y^2(1)} \quad \text{and} \quad w_2 = \frac{\rho_y(1)}{1 - \rho_y^2(1)}$$

where $\rho_y(1) = c_1/(1 + c_1^2)$. The predictor is therefore given as

$$\hat{y}_{3|y_1,y_2} = -\frac{c_1^2}{1 + c_1^2 + c_1^4} y_1 + \frac{c_1 + c_1^3}{1 + c_1^2 + c_1^4} y_2$$

Prediction of ARMA processes

We will now consider predicting an ARMA(p,q), such that

$$A(z)y_t = C(z)e_t$$

implying that

$$\begin{aligned} y_{t+k} &= \frac{C(z)}{A(z)}e_{t+k} = \sum_{\ell=0}^{\infty}\psi_\ell e_{t+k-\ell} \\ &= \sum_{\ell=0}^{k-1}\psi_\ell e_{t+k-\ell} + \sum_{\ell=k}^{\infty}\psi_\ell e_{t+k-\ell} \\ &= F(z)e_{t+k} + \sum_{\ell=k}^{\infty}\psi_\ell e_{t+k-\ell} \\ &= F(z)e_{t+k} + \sum_{\ell=0}^{\infty}\psi_\ell e_{t-\ell} \end{aligned}$$

where $F(z)$ is monic as $A(z)$ and $C(z)$ are. Thus,

$$F(z) = 1 + f_1 z^{-1} + \ldots + f_{k-1}z^{-k+1}$$

Prediction of ARMA processes

Proceeding, let

$$\begin{aligned} y_{t+k} &= F(z)e_{t+k} + \sum_{\ell=0}^{\infty}\psi_\ell e_{t-\ell} \\ &= F(z)e_{t+k} + \frac{G(z)}{A(z)}e_t \\ &= F(z)e_{t+k} + z^{-k}\frac{G(z)}{A(z)}e_{t+k} \end{aligned}$$

where the polynomials $G(z)$ and $F(z)$ satisfy the Diophantine equation

$$C(z) = A(z)F(z) + z^{-k}G(z)$$

with

$$\begin{aligned} &\text{ord}\,\{F(z)\} = k-1 \\ &\text{ord}\,\{G(z)\} = \max(p-1,\, q-k) \end{aligned}$$

with ord $\{F(z)\}$ denoting the order of the polynomial $F(z)$. Note that $G(z)$ is generally not monic.

Prediction of ARMA processes

The optimal linear prediction is formed as

$$\hat{y}_{t+k|t}(\boldsymbol{\Theta}) = E\{y_{t+k}|\boldsymbol{\Theta}\}$$

where

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\theta}^T & \mathbf{Y}_t^T \end{bmatrix}^T$$

with $\boldsymbol{\theta}$ denoting the model parameters and

$$\mathbf{Y}_t = \begin{bmatrix} y_1 & \ldots & y_t \end{bmatrix}^T$$

Thus,

$$\begin{aligned} \hat{y}_{t+k|t}(\boldsymbol{\Theta}) &= E\{y_{t+k}|\boldsymbol{\Theta}\} \\ &= E\Big\{F(z)e_{t+k}\Big|\boldsymbol{\Theta}\Big\} + E\Big\{z^{-k}\frac{G(z)}{A(z)}e_{t+k}\Big|\boldsymbol{\Theta}\Big\} \\ &= E\left\{z^{-k}\frac{G(z)}{A(z)}e_{t+k}\Big|\boldsymbol{\Theta}\right\} \\ &= E\left\{\frac{G(z)}{A(z)}\frac{A(z)}{C(z)}y_t\Big|\boldsymbol{\Theta}\right\} = \frac{G(z)}{C(z)}y_t \end{aligned}$$

Prediction of ARMA processes

The part that may not be predicted, the prediction error, may thus be written as

$$\begin{aligned} \epsilon_{t+k|t}(\boldsymbol{\Theta}) &= y_{t+k} - \hat{y}_{t+k}(\boldsymbol{\Theta}) \\ &= F(z)e_{t+k} + \frac{G(z)}{C(z)}y_t - \frac{G(z)}{C(z)}y_t \\ &= F(z)e_{t+k} \end{aligned}$$

implying that the prediction error should behave as an MA($k-1$) process, with

$$V\left\{\epsilon_{t+k|t}(\boldsymbol{\Theta})\right\} = \left(1 + f_1^2 + \ldots + f_{k-1}^2\right)\sigma_e^2$$

For a Normal distributed process, the $(1-\alpha)$ confidence prediction interval can therefore be expressed as

$$\hat{y}_{t+k|t} \pm u_{\alpha/2}\sigma_e\sqrt{1 + f_1^2 + \ldots + f_{k-1}^2}$$

where $u_{\alpha/2}$ denotes the $\alpha/2$ quantile in the standard Normal distribution.

Prediction of ARMA processes

Example: Compute the 5-step prediction of the SARIMA process

$$(1 - 0.2z^{-1})\nabla_{12} y_t = (1 - 0.3z^{-12})e_t$$

Thus,

$$A(z) = (1 - 0.2z^{-1})(1 - z^{-12}) = 1 - 0.2z^{-1} - z^{-12} + 0.2z^{-13}$$
$$C(z) = 1 - 0.3z^{-12}$$

implying that $p = \text{ord}\{A(z)\} = 13$ and $q = \text{ord}\{C(z)\} = 12$. For $k = 5$,

$$\text{ord}\{F(z)\} = 5 - 1 = 4$$
$$\text{ord}\{G(z)\} = \max(13 - 1,\ 12 - 5) = 12$$

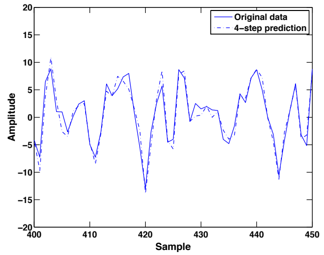
Performing the polynomial division yields

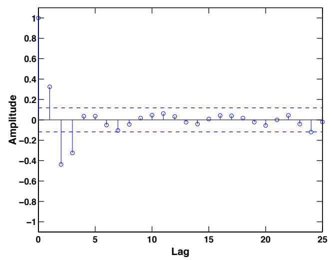
$$F(z) = 1 + 0.2z^{-1} + 0.2^2 z^{-2} + 0.2^3 z^{-3} + 0.2^4 z^{-4}$$
$$G(z) = 0.2^5 + 0.7z^{-7} - 0.2^5 z^{-12}$$

Thus, $(1 - 0.3z^{-12})\hat{y}_{t+5|t}(\boldsymbol{\Theta}) = (0.2^5 + 0.7z^{-7} - 0.2^5 z^{-12})y_t$, implying that

$$\hat{y}_{t+5|t}(\boldsymbol{\Theta}) = 0.2^5 y_t + y_{t-7} - 0.2^5 y_{t-12}$$

Prediction of ARMA processes





Example: Compute the 4-step prediction of the SARIMA process

$$(1 + 0.8z^{-1} + 0.8z^{-2})\nabla_{24} y_t = (1 + 0.4z^{-1} + 0.6z^{-14})e_t$$

For this process, $p = \text{ord}\{A(z)\} = 26$ and $q = \text{ord}\{C(z)\} = 14$. Thus, $\text{ord}\{F(z)\} = k - 1 = 3$ and $\text{ord}\{G(z)\} = \max(p-1,\ q-k) = 25$, yielding

$$F(z) = 1 - 0.4z^{-1} - 0.48z^{-2} + 0.704z^{-3}$$
$$G(z) = -0.1792 - 0.5632z^{-1} + 0.6z^{-10} + z^{-20} + 0.4z^{-21} + 0.1792z^{-24} + 0.5632z^{-25}$$

Note that the prediction error behaves like an MA(3).

Prediction of ARMAX processes

We proceed to the prediction of ARMAX processes, i.e.,

$$A(z)y_t = B(z)x_t + C(z)e_t$$

where

$$B(z) = b_d z^{-d} + b_{d+1} z^{-d-1} + \ldots + b_s z^{-s}$$

Then,

$$\begin{aligned} y_{t+k} &= \frac{C(z)}{C(z)} y_{t+k} \\ &= \frac{1}{C(z)} \Big\{ A(z)F(z) + z^{-k}G(z) \Big\} y_{t+k} \\ &= \frac{1}{C(z)} \Big\{ F(z)A(z)y_{t+k} + G(z)y_t \Big\} \end{aligned}$$

which yields

$$y_{t+k} = \frac{1}{C(z)} \Big\{ F(z)\Big[C(z)e_{t+k} + B(z)x_{t+k} \Big] + G(z)y_t \Big\}$$

Prediction of ARMAX processes

We proceed to rewrite

$$\frac{F(z)B(z)}{C(z)} x_{t+k} = \hat{F}(z)x_{t+k} + \frac{\hat{G}(z)}{C(z)} x_t$$

where the polynomials $\hat{F}(z)$ and $\hat{G}(z)$ are obtained by solving the corresponding Diophantine equation, i.e.,

$$F(z)B(z) = C(z)\hat{F}(z) + z^{-k}\hat{G}(z)$$

Thus,

$$\text{ord}\Big\{\hat{F}(z)\Big\} = k - 1$$
$$\text{ord}\Big\{\hat{G}(z)\Big\} = \max(q - 1,\ s - 1)$$

where we used that $\text{ord}\{F(z)B(z)\} = k - 1 + r$. This implies

$$\begin{aligned} y_{t+k} &= F(z)e_{t+k} + \frac{F(z)B(z)}{C(z)} x_{t+k} + \frac{G(z)}{C(z)} y_t \\ &= F(z)e_{t+k} + \hat{F}(z)x_{t+k} + \frac{\hat{G}(z)}{C(z)} x_t + \frac{G(z)}{C(z)} y_t \end{aligned}$$

Prediction of ARMAX processes

This gives the k-step prediction

$$\begin{aligned}\hat{y}_{t+k|t}(\boldsymbol{\Theta}) &= E\{y_{t+k}|\boldsymbol{\Theta}\}\\ &= F(z)E\{e_{t+k}|\boldsymbol{\Theta}\} + \hat{F}(z)E\{x_{t+k}|\boldsymbol{\Theta}\} + \frac{\hat{G}(z)}{C(z)}E\{x_t|\boldsymbol{\Theta}\} + \frac{G(z)}{C(z)}E\{y_t|\boldsymbol{\Theta}\}\\ &= \hat{F}(z)E\{x_{t+k}|\boldsymbol{\Theta}\} + \frac{\hat{G}(z)}{C(z)}x_t + \frac{G(z)}{C(z)}y_t\end{aligned}$$

Thus,

$$\epsilon_{t+k|t}(\boldsymbol{\Theta}) = F(z)e_{t+k} + \hat{F}(z)x_{t+k}$$

If e_t and x_t are independent,

$$\begin{aligned}V\{\epsilon_{t+k|t}(\boldsymbol{\Theta})\} &= V\{F(z)e_{t+k}\} + V\{\hat{F}(z)x_{t+k}|\boldsymbol{\Theta}\}\\ &= \sum_{\ell=0}^{k-1} f_\ell^2\sigma_e^2 + \sum_{\ell=0}^{k-1}\sum_{p=0}^{k-1}\hat{f}_\ell\hat{f}_p C\{x_{t+\ell}, x_{t+p}|\boldsymbol{\Theta}\}\end{aligned}$$