6 continued

To get a joint confidence interval of $0.95 = 1-\alpha$ we need α_1 so that

$$(1-\alpha_1)^2 = 0.95$$

so that

$$\alpha_1 = 0.025$$

and

$$z_{\alpha_1/2} = z_{0.0127} \simeq z_{0.01}$$

Finally

$$\hat{\theta} = \frac{1}{10} \sum x_i^2$$

$$Var(\hat{\theta}) = \frac{1}{10} Var(X_i^2)$$

$$= \frac{1}{10} \left(E(X_i^4) - (E(X_i^2))^2 \right)$$

for which the plug-in estimate is

$$\widehat{Var(\hat{\theta})} = \frac{1}{10} \left(\frac{1}{10} \sum x_i^4 - \left(\frac{1}{10} \sum x_i^2 \right)^2 \right)$$

$$= \ldots$$

Therefore

$$I_\eta \times I_\theta =$$

$$\left(\hat{\eta} \pm z_{0.01} \sqrt{\widehat{Var(\hat{\eta})}} \right) \times \left(\hat{\theta} \pm z_{0.01} \sqrt{\widehat{Var(\hat{\theta})}} \right)$$

is a joint CI with approximate simult. confidence grade 0.95.

6. We have

$$\theta = \int x^2 \, dF(x)$$
$$\eta = \int y \, dG(y)$$

a) $\hat{\theta} = \frac{1}{10} \sum x_i^2 = \ldots$

$\hat{\eta} = \frac{1}{10} \sum y_i = \ldots$

$\widehat{(\theta - \eta)} = \hat{\theta} - \hat{\eta} = \ldots$

b) $Var(\hat{\eta}) = \frac{1}{10} Var(Y_1)$

$= \frac{1}{10} \left(E(Y_1^2) - (E(Y_1))^2 \right)$

for which the plug-in estimator is

$\widehat{Var(\hat{\eta})} = \frac{1}{10} \left(\frac{1}{10} \sum y_i^2 - \left(\frac{1}{10} \sum y_i \right)^2 \right) = \ldots$

CLT + LLN + Slutsky $\Rightarrow$

$$I_\eta = \left(\hat{\eta} \pm 1.96 \cdot \sqrt{\widehat{Var(\hat{\eta})}} \right)$$

c) A $1-\alpha_1$ confidence interval for η is

$$I_\eta = \left(\hat{\eta} \pm z_{\alpha_1/2} \sqrt{\widehat{Var(\hat{\eta})}} \right)$$

and for θ is

$$I_\theta = \left(\hat{\theta} \pm z_{\alpha_1/2} \sqrt{\widehat{Var(\hat{\theta})}} \right)$$

5 (continued)

$$\mathrm{Var}(\hat{y}) = \mathrm{Var}(\hat{\beta}_0) + a^2\,\mathrm{Var}(\hat{\beta}_1) + b^2\,\mathrm{Var}(\hat{\beta}_2) + 2a\,\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) + 2b\,\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1) + 2ab\,\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_2)$$

The values of the variances and covariances are picked from the matrix $(X^T X)^{-1}\sigma^2$

The confidence interval is therefore

$$\cancel{I = (\hat{y} \pm z_{0.025} \cdot \sigma^2 \mathrm{Var}}$$

$$I = \left(\hat{y} \pm z_{0.025} \sqrt{\mathrm{Var}(\hat{y})}\right)$$

5. $Y = X\beta + \varepsilon$ is the matrix formulation of the regression problem.

$\sigma^2 = 0.25$

a) $\hat{\beta} = (X^T X)^{-1}(X^T Y)$

$$= \begin{pmatrix} 0.5759 \\ 0.1295 \\ 0.6513 \end{pmatrix}$$

We know that

$$\hat{\beta} \in N_3(\beta, \sigma^2 (X^T X)^{-1})$$

so that

$$Var(\hat{\beta}_1) = 0.25 \cdot 0.1226$$

so a test statistic is

$$t = \frac{\hat{\beta}_1}{\sqrt{0.25 \cdot 0.1226}} = 0.7387$$

Compare this with for instance $z_{0.025} = 1.96$

Conclusion, no not significant

b) The expected total delay is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \underbrace{(0.5 - 0.5315282)}_{a} + \hat{\beta}_2 \underbrace{(0.4 - 0.4853)}_{b}$$

which has expectation

$$E(\hat{y}) = \beta_0 + \beta_1 a + \beta_2 b$$

and variance

4. $X = X(t)$ the number of particles that decay during t seconds is $Po(\theta t)$-distributed.
$E(X) = \theta t$

a) Least squares estimator is

$$\hat{\theta} = \arg\min_{\theta} (26 - \theta \cdot 5)^2 = \frac{26}{5} = 5.2$$

b) Want to test

$$H_0: \theta \leq 3$$
$$H_1: \theta > 3$$

Makes sense to reject for large values of X, so p-value method gives

$$p = P(X \geq 26 \mid \theta = 3) = P(X \geq 26 \mid X \in Po(15))$$
$$= 1 - P(X \leq 25 \mid X \in Po(15))$$
$$(\text{table})$$
$$= 1 - 0.9938 = 0.0062$$

Reject H_0 on level 0.0062.

3 $\bar{X}_A \in N(\mu_A, \sigma^2)$

$X_B \in N(\mu_B, \sigma^2)$ $\quad X_A, X_B$ independent

$x_1^A, \dots, x_8^A$ obs. of X_A

$x_1^B, \dots, x_7^B$ obs. of X_B

The confidence interval for $\mu_A - \mu_B$ is given by

$$I_{\mu_A - \mu_B} = (\bar{x}^A - \bar{x}^B \pm t_{0.025}(f)\, d)$$

where

$$f = 8 - 1 + 7 - 1 = 13$$

$$d = \sqrt{\frac{s_p^2}{8} + \frac{s_p^2}{7}} = s_p \sqrt{\frac{1}{8} + \frac{1}{7}}$$

$$s_p^2 = \frac{(8-1)s_{x_A}^2 + (7-1)s_{x_B}^2}{(8-1)+(7-1)}$$

and

$$s_{x_A}^2 = \left(\sum (x_i^A)^2 - 8(\bar{x}^A)^2\right)/(8-1)$$

$$s_{x_B}^2 = \left(\sum (x_i^B)^2 - 8(\bar{x}^B)^2\right)/(7-1)$$

$\bar{x}^A = \dots$ $\quad \sum (x_i^A)^2 = \dots$

$\bar{x}^B = \dots$ $\quad \sum (x_i^B)^2 = \dots$

$s_p^2 = \dots$ $\quad t_{0.025}(13) = 2.16$

$I_{\mu_A - \mu_B} (\dots \pm \dots)$

2. If X is the number of trials until success (including the last one) and $\theta = P(\text{success})$ then

$$P(X=k) = f_\theta(k) = (1-\theta)^{k-1}\theta, \quad k=1,2,3,\dots$$

The likelihood is

$$L(\theta) = f_\theta(3) = (1-\theta)^{3-1}\theta = (1-\theta)^2\theta$$

and

$$\ell(\theta) = \log L(\theta) = 2\log(1-\theta) + \log\theta$$

$$\ell'(\theta) = -\frac{2}{1-\theta} + \frac{1}{\theta} = 0$$

$\Leftrightarrow$ if $\theta \neq 0$ $\quad 0 = -2\theta + 1 - \theta = -3\theta + 1$

$\Leftrightarrow \quad \theta = \frac{1}{3}$

so MLE is $\hat{\theta} = \frac{1}{3}$

In general, if x is the observation the MLE is $\hat{\theta} = \frac{1}{x}$, so large values of x point to small values of θ and vice versa. Therefore reject H_0 if x is large enough. The p-value method gives

$$p = P(X \geq 3 \mid \theta = 1/25) = 1 - P(X \leq 2 \mid \theta = \tfrac{1}{25})$$

$$= 1 - \left(P(X=1 \mid \theta = \tfrac{1}{25}) + P(X=2 \mid \theta = \tfrac{1}{25})\right)$$

$$= 1 - \left(\left(1-\tfrac{1}{25}\right)^1 \tfrac{1}{25} + \left(1-\tfrac{1}{25}\right)^0 \tfrac{1}{25}\right)$$

$$= 0.92$$

and we do not reject H_0

(If we would have tested

$H_0: \theta \leq 1/25$

$H_1: \theta > 1/25$

p-value would have been 0.08)

1. X cont r.v. with prob. density function

$$f_\theta(x) = \begin{cases} \frac{1}{2\theta} & -\theta \le x \le 0 \\ \frac{1}{2\theta} e^{-x/\theta} & x \ge 0 \end{cases}$$

Observations are $-2.3, -4, 1.4, -5.1, 1.7$.

The likelihood function is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{5} f_\theta(x_i) \\ &= \frac{1}{2\theta} 1\{-\theta \le -2.3\} \cdot \frac{1}{2\theta} 1\{-\theta \le -4\} \\ &\quad \cdot \frac{1}{2\theta} e^{-1.4/\theta} \cdot \frac{1}{2\theta} 1\{-\theta \le -5.1\} \cdot \frac{1}{2\theta} e^{-1.7/\theta} \\ &= \frac{1}{(2\theta)^5} e^{-3.1/\theta} 1\{\theta \ge 2.3\} \cdot 1\{\theta \ge 4\} \cdot 1\{\theta \ge 5.1\} \\ &= \frac{1}{(2\theta)^5} e^{-3.1/\theta} \cdot 1\{\theta \ge 5.1\} \end{aligned}$$

To maximize this, we this as the problem of maximizing the function

$$\tilde{L}(\theta) = \frac{1}{(2\theta)^5} e^{-3.1/\theta}$$

over the set $[5.1, \infty)$. Then

$$\tilde{\ell}(\theta) = \log \tilde{L}(\theta) = -5\log(2\theta) - \frac{1}{\theta} 3.1$$

and

$$\tilde{\ell}'(\theta) = 0 = -\frac{5}{\theta} + \frac{3.1}{\theta^2}$$

$$\iff -5\theta + 3.1 = 0$$

if $\theta \neq 0$ $\iff$ $\theta = \frac{3.1}{5} = 0.62$

This is not an interior point of $[5.1, \infty)$ and $\tilde{\ell}'(\theta) < 0$ for $\theta > 0.62$ so $\tilde{\ell} \downarrow$ so maximum is attained at boundary and therefore

$$\underline{\hat{\theta} = 5.1}$$