6 continued

To get a joint confidence interval of 0.95 = 1 - α
we need α₁ so that
\[(1 - α₁)² = 0.95\]
so that
\[α₁ = 0.025\]
and
\[2α₁/2 = 2 × 0.0125 = 0.01\]

Finally

\[θ_{\bar{x}_1} = \frac{1}{10} \sum x_i^2\]
\[\text{Var}(θ) = \frac{1}{10} \text{Var}(x_i^2)\]
\[= \frac{1}{10} \left( E(x_i^4) - (E(x_i^2))^2 \right)\]

for which the plug-in estimate is
\[\hat{θ} = \frac{1}{10} \left( \frac{1}{10} \sum x_i^4 - \left( \frac{1}{10} \sum x_i^2 \right)^2 \right)\]

Therefore

\[\left( \bar{x}_1 \pm 2.01 \sqrt{\frac{\text{Var}(θ)}{n}} \right) × \left( \bar{x}_1 \pm 2.01 \sqrt{\text{Var}(θ)} \right)\]

is a joint CI with approximate simultaneity confidence grade 0.95.
6. We have

\[ \theta = \int x^2 \, f(x) \, dx \]
\[ \nu = \int_{-\infty}^{\infty} \theta \, f(x) \, dx \]

a) \[ \bar{\theta} = \frac{1}{10} \sum x_i^2 = \ldots \]
\[ \bar{\nu} = \frac{1}{10} \sum y_i = \ldots \]
\[ \bar{\theta} - \bar{\nu} = \bar{\theta} - \bar{\nu} = \ldots \]

b) \[ \text{Var}(\bar{\nu}) = \frac{1}{10} \, \text{Var}(Y) \]
\[ = \frac{1}{10} \left( \text{Var}(Y) - (\text{Var}(Y))^2 \right) \]

For which the plug-in estimator \( \bar{\nu} \)
\[ \frac{1}{\text{Var}(\bar{\nu})} = \frac{1}{10} \left( \frac{1}{10} \sum y_i^2 - (\frac{1}{10} \sum y_i)^2 \right) = \ldots \]

\[ \text{C.I.} + 2.24 N + \text{estimation} \Rightarrow \overline{\nu} \]

\[ \text{I}_{\nu} = (\bar{\nu} \pm 1.96 \cdot \sqrt{\text{Var}(\bar{\nu})}) \]

C) A \( 1 - \alpha \) confidence interval for \( \theta \)
\[ \text{I}_{\theta} = (\bar{\theta} \pm z_{\alpha / 2} \sqrt{\text{Var}(\bar{\theta})}) \]

and for \( \theta \) is
\[ \text{I}_{\theta} = (\bar{\theta} \pm z_{\alpha / 2} \sqrt{\text{Var}(\bar{\theta})}) \]
\( \text{Var}(\hat{\gamma}) = \text{Var}(\hat{\beta}_0) + a^2 \text{Var}(\hat{\beta}_1) + b^2 \text{Var}(\hat{\beta}_2) \\
+ 2a \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) + 2b \text{Cov}(\hat{\beta}_0, \hat{\beta}_2) \\
+ 2ab \text{Cov}(\hat{\beta}_1, \hat{\beta}_2) \)

The values of the variances and covariances are picked from the matrix \((X'X)^{-1}\) of \(\sigma^2\).

The confidence interval is therefore

\[ I = (\hat{\gamma}_0 \pm 2.095 \sqrt{\text{Var}(\hat{\gamma})}) \]
5. \( Y = X\beta + \epsilon \) is the matrix formulation of the regression problem.
\[ \sigma^2 = 0.25 \]

\[ \beta = (X^T X)^{-1} (X^T Y) \]
\[ = \begin{pmatrix} 0.5759 \\ 0.1295 \\ 0.6135 \end{pmatrix} \]

We know that
\[ \hat{\beta} \sim N_3 \left( \beta, \sigma^2 (X^T X)^{-1} \right) \]
so that
\[ \text{Var}(\hat{\beta}_1) = 0.25 \cdot 0.1226 \]
So a test statistic is
\[ t = \frac{\hat{\beta}_1}{\sqrt{0.25 \cdot 0.1226}} = 0.7367 \]

Compare this with for instance \( t_{0.025, 1} = 1.96 \)
Conclusion, is not significant

b) The expected total delay is
\[ \bar{Y} = \frac{1}{6} (\beta_0 + \beta_1 \left( 0.5 - 0.53 \frac{15 - 2F}{2} \right) + \beta_2 \left( 0.5 - 0.48 \frac{15 - 0.5}{2} \right) \]

which has expectation
\[ E(\bar{Y}) = \beta_0 + \beta_1 + \beta_2 \]
and variance
4. \( X = \mathbb{E}(t) \) the number of particles that
decay during \( t \) seconds is \( \text{Po}(\theta t) \)-distributed.
\( \mathbb{E}(X) = \theta t \)

a) Least squares estimator is
\[ \hat{\theta} = \arg \min_{\theta} \left( \frac{26 - \theta \cdot 5}{5} \right)^2 = \frac{26}{5} = 5.2 \]

b) Want to test
\[ H_0 : \theta \leq 3 \]
\[ H_1 : \theta > 3 \]
Makes sense to reject for large values of \( \bar{X} \).
so \( p \)-value method gives
\[ p = P(\bar{X} \geq 26 | \theta = 3) = P(\bar{X} \geq 26 | \bar{X} \sim \text{Po}(15)) \]
\[ = 1 - P(\bar{X} \leq 25 | \bar{X} \sim \text{Po}(15)) \]
( table)
\[ = 1 - 0.99382 = 0.0062 \]
Reject \( H_0 \) at level 0.0062.
\[ \bar{x}_A \sim N(\mu_A, \sigma^2) \]
\[ \bar{x}_B \sim N(\mu_B, \sigma^2) \]
\[ \text{independent} \quad \bar{x}_A / \bar{x}_B \]
\[ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \quad \text{iid of } \bar{x}_A \]
\[ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \quad \text{iid of } \bar{x}_B \]

The confidence interval for \( \mu_A - \mu_B \) is

\[ n_{\text{a}} \quad b_{\alpha} \]

\[ I_{\mu_A - \mu_B} = (\bar{x}_A - \bar{x}_B \pm t_{0.025}(f) \cdot d) \]

where

\[ f = \frac{\delta - 1}{\sqrt{\frac{s^2_A}{n_A} + \frac{s^2_B}{n_B}}} = \sqrt{\frac{s^2_A}{s^2_A + s^2_B}} \]

\[ d = \frac{1}{\delta} + \frac{1}{f} \]

\[ \bar{x}_A - \bar{x}_B = \frac{(f-1)s^2_A + (f-1)s^2_B}{f + (f-1)} \]

and

\[ s^2_{x_A} = \frac{2}{n_A} \sum (x_i - \bar{x}_A)^2 \]

\[ s^2_{x_B} = \frac{2}{n_B} \sum (x_i - \bar{x}_B)^2 \]

\[ \bar{x}_A = \ldots \quad \bar{x}_B = \ldots \]

\[ s^2_{x_A} = \ldots \quad s^2_{x_B} = \ldots \]

\[ t_{0.025}(13) = 2.16 \]

\[ I_{\mu_A - \mu_B} (\ldots \pm \ldots \ldots) \]
2. If $X$ is the number of trials with success (including the last one) and $\theta = P(\text{success})$
then
$$P(X = k) = f_\theta(k) = (1 - \theta)^{k-1} \theta, \quad k = 1, 2, 3, \ldots$$

The likelihood is
$$L(\theta) = f_\theta(3) = (1 - \theta)^{3-1} \theta = (1 - \theta)^2 \theta$$
and
$$l(\theta) = \log_2 L(\theta) = 2 \log_2 (1 - \theta) + \log_2 \theta$$
$$l'(\theta) = -\frac{2}{1 - \theta} + \frac{1}{\theta} = 0$$

If $l'(\theta) = 0$, then $\theta = \frac{1}{3}$.

So $\hat{\theta}$ is the MLE with $\hat{\theta} = \frac{1}{3}$.

In general, $X$ is the observation that MLE is $\hat{\theta} = \frac{1}{X}$ for large values of $X$ and small values of $\theta$ and vice versa. Therefore reject $H_0$ if $X$ is large enough. The $p$-value method gives
$$p = P(X \geq 3 \mid \theta = \frac{1}{25}) = P(X \leq 2 \mid \theta = \frac{1}{25})$$
$$= 1 - P(X = 1 \mid \theta = \frac{1}{25}) = P(X = 2 \mid \theta = \frac{1}{25})$$
$$= 1 - \left(1 - \frac{1}{25}\right) \left(\frac{1}{25}\right) + \left(1 - \frac{1}{25}\right)\left(0\right) \left(\frac{1}{25}\right)$$
$$= 0.02$$

and we do not reject $H_0$.

If we would have tested
$$H_0 : \theta = \frac{1}{25}$$
$$H_1 : \theta > \frac{1}{25}$$
the $p$-value would have been $0.02$. 


1. I cut rv. with prob. density function

\[ f_\theta(x) = \begin{cases} \frac{1}{2\theta} & -\theta \leq x \leq 0 \\ \frac{1}{2\theta} e^{-x/\theta} & x \geq 0 \end{cases} \]

Observations are \(-2.7, -4.3, 4.4, -5.1, 3.7\).

The likelihood function is

\[ L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) \]

\[ = \frac{1}{(2\theta)^n} \left( e^{-\sum_{i=1}^{n} x_i/\theta} \right) \]

\[ = \frac{1}{(2\theta)^n} e^{-\sum_{i=1}^{n} x_i/\theta} \]

To maximize this, we need to solve the problem of maximizing the function

\[ L(\theta) = \frac{1}{(2\theta)^5} e^{-3.1/\theta} \]

over the set \([5.1, \infty)\). Then

\[ \hat{L}(\theta) = \log L(\theta) = -5 \log(2\theta) - \frac{1.71}{\theta} \]

and

\[ \hat{\theta}'(\theta) = 0 = -\frac{5}{\theta} + \frac{1.71}{\theta^2} \]

if \(\theta \neq 0 \Rightarrow \theta = \frac{1.71}{5} = 0.34 \]

This is not an interior point of \([5.1, \infty)\) and \(\hat{\theta}'(\theta) < 0\) for \(\theta > 0.34\) so the maximum is attained at boundary and therefore

\[ \hat{\theta} = 5.1 \]