

1 a) Yes, since $\{\omega: X(\omega)=k\} \in \mathcal{F}$
for $k=0$ and 1 .

b) The probability mass function is

$$f_X(k) = P(X=k)$$

for $k=1, 2, \dots, 6$. Thus

~~$$f_X(k) = P(X=k)$$~~

$$f_X(0) = P(X=0) =$$

$$= P(\omega: X(\omega)=0) =$$

$$= P(2) + P(4) + P(6) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$f_X(1) = P(\omega: X(\omega)=1) =$$

$$= P(1) + P(3) + P(5) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

- $P(H|G) = 0.1$
- $P(H) = 0.3$
- $P(G) = 0.5$
- $A = \text{[version]}$
- $P(A|H) = 0.3$
- $P(A|G) = 0.1$
- $P(A|G) = 0.05$

a) $P(H|A) = \frac{P(A \cap H)}{P(A)}$

$$= \frac{P(A|H)P(H) + P(A|G)P(G)}{P(A|H)P(H) + P(A|G)P(G)}$$

$$= \frac{0.3 \cdot 0.1 + 0.1 \cdot 0.3 + 0.6 \cdot 0.05}{0.3 \cdot 0.1 + 0.1 \cdot 0.3 + 0.6 \cdot 0.05} = \frac{0.08}{0.08} = 1$$

b) $P(A) = P(A|H)P(H) + P(A|G)P(G)$

$$= 0.3 \cdot 0.1 + 0.1 \cdot 0.3 + 0.6 \cdot 0.05 = 0.09$$

$P(\text{at least one error}) = 1 - P(\text{no one error})$

$$= 1 - P(A^c \cap \dots \cap A^c) = 1 - P(A^c)^N = 1 - (1 - 0.09)^N \geq 0.9$$

$$N \geq \frac{50}{0.05} \ln(1 - 0.05)$$

$$\ln(1 - 0.05) =$$

$$N \geq \frac{50 \ln(1 - 0.05)}{\log 0.1} \Rightarrow$$

$$1.0 \log 0.1 \leq (1 - 0.05) \cdot N \Rightarrow$$

$$1.0 \leq N(1 - 0.05) \Rightarrow$$

2) cont.

$$(1 - 0.05)^N \geq 0.05$$

3

a)

$$X_1 \in \text{Bin}(4, 0.5)$$

$$X_2 \in \text{Bin}(1, 0.5) = \text{Bern}(1, 0.5)$$

X_1, X_2 independent \Rightarrow

$$X_1 + X_2 \in \text{Bin}(5, 0.5)$$

b)

$$P_X(k) = \binom{5}{k} 0.5^k 0.5^{5-k}$$

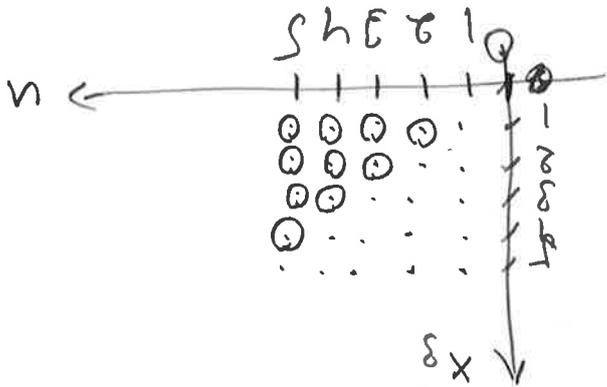
$$= \binom{5}{k} 0.5^5, \quad k = 0, 1, 2, \dots, 5$$

b)

$$U = X_1 + X_2 \in \text{Bin}(5, 0.5) \quad \text{take value } 0, 1, \dots, 5$$

$$X_3 \in \text{Ge}(0.1) \quad \text{take value } 1, 2, 3, \dots$$

$$P(U > X_3)$$



$$P(U > X_3) = P((U, X_3) \in (2,1) \cup (3,1) \cup (4,1) \cup (5,1) \cup (3,2) \cup (4,2) \cup (5,2) \cup (4,3) \cup (5,3) \cup (5,4))$$

3rd

Since they are independent this becomes

$$\begin{aligned}
& \binom{5}{2} 0.5^2 \cdot 0.1 + \binom{5}{3} 0.5^3 \cdot 0.1 + \binom{5}{4} 0.5^4 \cdot 0.1 + \binom{5}{5} 0.5^5 \cdot 0.1 \\
& + \binom{5}{2} 0.5^2 \cdot 0.9 \cdot 0.1 + \binom{5}{3} 0.5^3 \cdot 0.9 \cdot 0.1 + \binom{5}{4} 0.5^4 \cdot 0.9 \cdot 0.1 + \binom{5}{5} 0.5^5 \cdot 0.9 \cdot 0.1 + \dots \\
& + \binom{5}{2} 0.5^2 \cdot 0.9^2 \cdot 0.1 + \dots \\
& + \binom{5}{5} 0.5^5 \cdot 0.9^4 \cdot 0.1 = \dots
\end{aligned}$$

X_1, X_2, \dots, X_n i.i.d. $U[0,1]$ -distributed r.v.'s.

$$Y_1 = \lfloor n X_1 \rfloor \leq \lfloor n Y \rfloor$$

\Rightarrow

$$Y_1 \in \text{Bern}(p) \text{ with } p = P(Y_1 = 1) = P(X_1 \leq \frac{1}{n}) = \frac{1}{n}$$

so that

$$Y_1 \in \text{Bern}(\frac{1}{n})$$

a) $S_{10} = Y_1 + \dots + Y_{10}$ a sum of 10 independent

$\text{Bern}(\frac{1}{n})$ -distributed r.v.'s, thus

$$S_{10} \in \text{Bin}(10, \frac{1}{n})$$

$$P(S_{10} = 4) = \binom{10}{4} \left(\frac{1}{n}\right)^4 \left(1 - \frac{1}{n}\right)^{10-4} \quad \text{or from}$$

$$\text{table} = F_{S_{10}}(4) - F_{S_{10}}(3) =$$

$$= 0.92187 - 0.77588$$

$$= 0.1460$$

5) By the central limit theorem we

have that for large n

$$Y_1 + \dots + Y_n = S_n$$

is approximately distributed as a Normal (Gaussian) random variable. Also

$$E(S_n) = 100 \cdot \frac{1}{4} = 25$$

$$Var(S_{100}) = 100 \cdot \frac{1}{3} \cdot \frac{1}{4} = 18.75$$

Therefore

$$P(21 < S_{100} \leq 29) = P\left(\frac{21-25}{\sqrt{18.75}} < Z \leq \frac{29-25}{\sqrt{18.75}}\right)$$

$Z \in N(0,1)$

$$\approx \Phi\left(\frac{29-25}{\sqrt{18.75}}\right) - \Phi\left(\frac{21-25}{\sqrt{18.75}}\right)$$

$$= \Phi(0.9238) - \Phi(-0.9238)$$

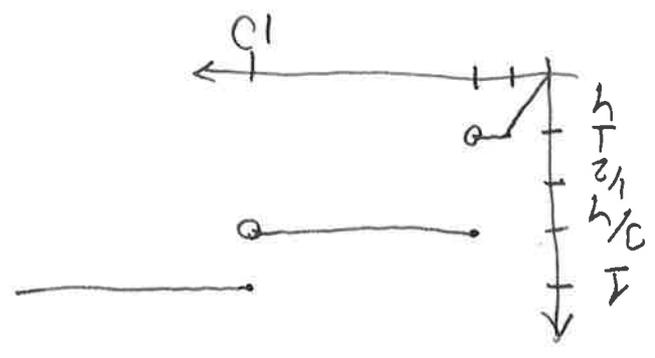
table 1.

$$\approx 0.82 - (1 - 0.82) = 0.64$$

with respect to $\mu(x) = \mu(x)1\{x < 1\} + \mu(x)1\{x \geq 1\}$, where μ is length measure and μ_1 is counting measure.

$$f(u) = \begin{cases} 0 & u < 0 \\ 0 & 0 \leq u < 1 \\ \frac{1}{2} & 1 \leq u < 2 \\ \frac{1}{4} & 2 \leq u < 10 \\ 0 & u \geq 10 \end{cases}$$

The density is



a)

$$F(u) = \begin{cases} 0 & u < 0 \\ 0 & 0 \leq u < 1 \\ \frac{1}{2} & 1 \leq u < 2 \\ \frac{3}{4} & 2 \leq u < 10 \\ 1 & u \geq 10 \end{cases}$$

5. U n.v. with d.f.



$$P(-\log(1-u) \geq y) =$$

$$P(-\log(1-u) \leq y) =$$

in which case $Y = -\log(1-u)$

$$P(0 < Y \leq y) = P(0 < -\log(1-u) \leq y) \text{ when } 0 \leq u < 1$$

$$\overline{0 < Y \leq y}$$

$$= 1 - F_U(1 - e^{-y}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(Y = y) = P(U \geq 1) = 1 - P(U < 1)$$

$$\overline{Y = 0}$$

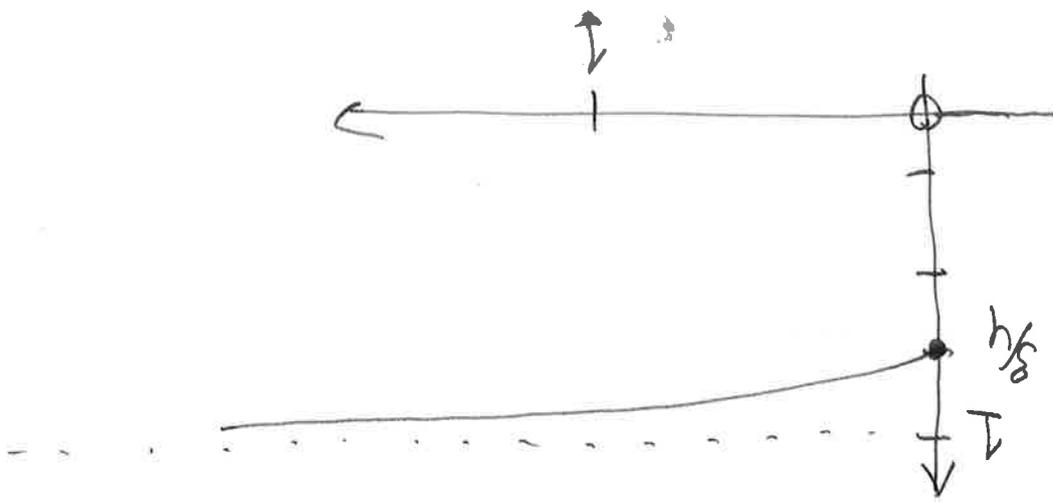
$Y = 0$ and $0 < Y \leq y$ (for some $y > 0$)

We can distinguish between two cases

that Y takes values in $[0, \infty)$.

From the definition of Y , we see

$$Y = g(U) = \begin{cases} 0 & 0 \leq U < 1 \\ -\log(1-u) & U \geq 1 \end{cases}$$



$$= \frac{y}{8} + \frac{1}{4}(1 - e^{-y}) \quad \text{if } y > 0$$

$$F_Y(y) = P(Y \leq y) = P(Y = 0) + P(0 < Y \leq y)$$

and

$$F_Y(y) = \frac{y}{8} \quad \text{if } y = 0$$

Therefore

$$P(Y = 0) = \frac{1}{4}(1 - e^{-1})$$

$$= F_Y(1 - e^{-1})$$

$$= P(1 - e^{-Y} \leq 1 - e^{-1}) = P(Y \geq 1) \quad (\text{when } Y > 0 \text{ then } 1 - e^{-Y} < 1)$$

$$= P(1 - e^{-Y} \geq e^{-1})$$

5 y) the

6. N = the number of travellers that arrive at the train station during a day

$N \sim \text{Bin}(3, 0.4)$

T_1, T_2, \dots, T_N = the waiting times

for the N travellers

$T = \max(T_1, \dots, T_N)$, T_1, \dots, T_N iid. $\text{Exp}(1/10)$
 T_1, \dots, T_N independent of N .

a) $P(T > 5) = \sum_{k=0}^3 P(T > 5 | N=k) \cdot P(N=k)$

$= \sum_{k=0}^3 P(\min(T_1, \dots, T_k) > 5 | N=k) P(N=k)$

$= \sum_{k=0}^3 P(\min(T_1, \dots, T_k) > 5) \binom{3}{k} 0.4^k 0.6^{3-k}$

it's independent of N

$P(\min(T_1, \dots, T_k) > 5) = P(T_1 > 5, \dots, T_k > 5)$
 $= P(T_1 > 5) \cdot \dots \cdot P(T_k > 5)$
 $= e^{-5/10} \cdot \dots \cdot e^{-5/10} = e^{-k \cdot 5/10}$

so then $E(T | N=k) = \frac{10}{k}$

$$F_T(t) = 1 - e^{-kt/10}$$

That then

if $N=k$ then T is the minimum of k indep. exp distributed r.v. and we know, or have shown in c)

$$E(T) = \sum_{k=1}^{\infty} E(T | N=k) \cdot P(N=k)$$

b) $E(T) = E(E(T | N))$

$$= e^{-5/10} \binom{3}{1} 0.4 \cdot 0.6 + e^{-10/10} \binom{3}{2} 0.4^2 \cdot 0.6 + e^{-15/10} \binom{3}{3} 0.4^3 = 0.3823$$

$$P(T > 5) = e^{-5/10} \binom{3}{3} 0.4 \cdot 0.6^3$$

6.90

6th Thus we obtain

$$E(T) = \sum_{k=0}^3 E(T|N=k) P(N=k)$$

$$= \sum_{k=1}^3 \frac{10}{k} \cdot \binom{3}{k} 0.4^k 0.6^{3-k}$$

$$= 10 \cdot \binom{3}{1} 0.4 \cdot 0.6^2$$

$$+ \frac{10}{2} \binom{3}{2} 0.4^2 \cdot 0.6$$

$$+ \frac{10}{3} \binom{3}{3} 0.4^3 \cdot 1 = 5.9733$$