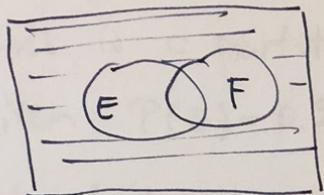


1.  $(\Omega, \mathcal{F}, P)$  probability space,  $E, F$  events.  
 $P(E) = 0.2, P(F) = 0.3, P(E \cap F) = 0.1$



$$\begin{aligned}
 P((E \cup F)^c) &= 1 - P(E \cup F) \\
 &= 1 - (P(E) + P(F) - P(E \cap F)) \\
 &= 1 - (0.2 + 0.3 - 0.1) = \underline{0.6}
 \end{aligned}$$

a)  $P(A) = P(A|E)P(E) + P(A|F)P(F) + P(A|\bar{E}\bar{F})P(\bar{E}\bar{F})$   
 $= 0.1 \cdot 0.2 + 0.5 \cdot 0.3 + 0.3 \cdot 0.4$   
 $= 0.34$  (by law of total probability)

b)  $P(F|A) = \frac{P(F \cap A)}{P(A)} = \frac{P(A|F)P(F)}{P(A)}$

= (which is Bayes formula)

$$= \frac{0.5 \cdot 0.4}{0.34} = \frac{0.2}{0.34} = \frac{10}{17} = 0.5882$$

2. Birds can be in three different states

$$E, F, S$$

and  $\Omega = E \cup F \cup S$ , and they are disjoint  $\Rightarrow$   
this is a partition.

Given  $P(E) = 0.2, P(F) = 0.4, P(S) = 0.4$

Let  $A$  be the event that a bird is caught in  
a net. Then we are also given

$$P(A|E) = 0.1, P(A|F) = 0.5, P(A|S) = 0.3$$

a) 
$$\begin{aligned} P(A) &= P(A|E)P(E) + P(A|F)P(F) + P(A|S)P(S) \\ &= 0.1 \cdot 0.2 + 0.5 \cdot 0.4 + 0.3 \cdot 0.4 \\ &= 0.34 \quad (\text{by law of total probability}) \end{aligned}$$

b) 
$$P(F|A) = \frac{P(F \cap A)}{P(A)} = \frac{P(A|F)P(F)}{P(A)} =$$

= (which is Bayes formula)

$$= \frac{0.5 \cdot 0.4}{0.34} = \frac{0.2}{0.34} = \frac{10}{17} = 0.5882$$

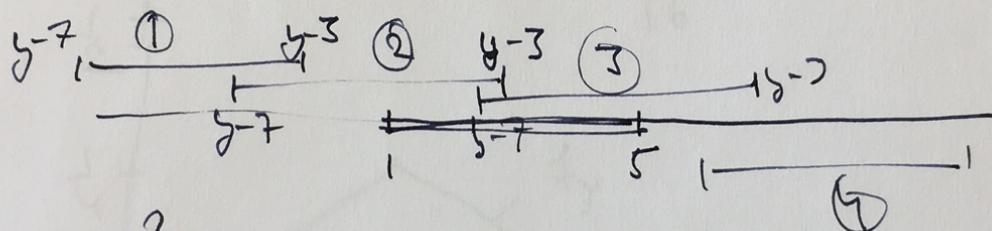
3.  $X_1, X_2$  independent  $\text{Un}[1, 5]$  and  $\text{Un}[3, 7]$  respectively.

a) The convolution formula says that, if  $Y = X_1 + X_2$ ,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(u) f_{X_2}(y-u) du$$

$$f_{X_1}(u) = \begin{cases} \frac{1}{4} & 1 \leq u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X_2}(y-u) = \begin{cases} \frac{1}{4} & 3 \leq y-u \leq 7 \Leftrightarrow y-7 \leq u \leq y-3 \\ 0 & \text{otherwise} \end{cases}$$



$$\textcircled{1} \Leftrightarrow \left. \begin{array}{l} y-3 \leq u \\ y \leq u \end{array} \right\} \Rightarrow f_Y(y) = 0 \quad \text{since the two functions } f_{X_1}(u), f_{X_2}(y-u) \text{ are then } \underline{\text{not}} \text{ different from zero for any } y \leq u.$$

$$\textcircled{4} \Leftrightarrow \left. \begin{array}{l} y-7 \geq u \\ y \geq u \end{array} \right\} \Rightarrow f_Y(y) = 0 \quad -11-$$

3 (c)

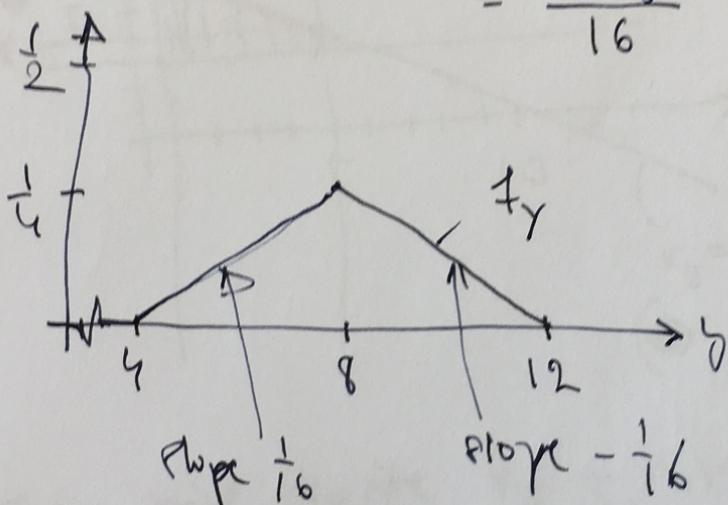
a)

②  $\Leftrightarrow \begin{cases} 7 \leq Y \leq 8 \\ 4 \leq Y \leq 8 \end{cases} \Rightarrow f_Y(y) = \int_4^y \frac{1}{4} \cdot \frac{1}{4} dy = \frac{y-4}{16}$

$$= \frac{y-4}{16}$$

③  $\Leftrightarrow \begin{cases} 7 \leq Y \leq 8 \\ 8 \leq Y \leq 12 \end{cases} \Rightarrow f_Y(y) = \int_{y-7}^8 \frac{1}{4} \cdot \frac{1}{4} dy = \frac{8-(y-7)}{16}$

$$= \frac{15-y}{16}$$



Thus:

$$f_Y(y) = \begin{cases} 0 & y \leq 4 \\ \frac{y-4}{16} & 4 \leq y \leq 8 \\ \frac{15-y}{16} & 8 \leq y \leq 12 \\ 0 & y \geq 12 \end{cases}$$

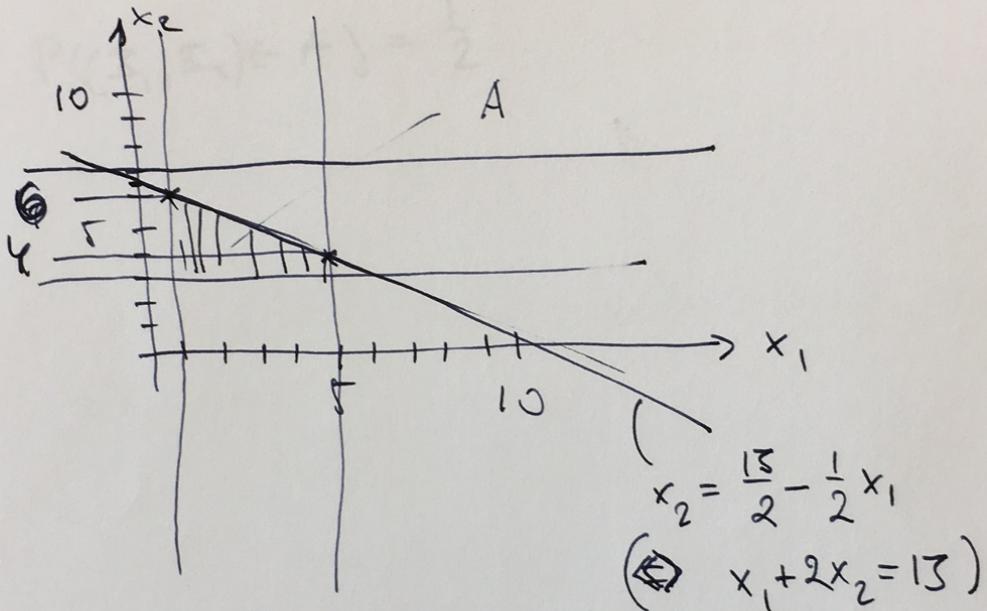
3 (cont.)

b)  $P(X_1 + 2X_2 \leq 13) = P((X_1, X_2) \in A)$

with  $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 \leq 13\}$ .

and

$$P((X_1, X_2) \in A) = \iint_A f_{X_1, X_2}(x_1, x_2) dx_1 dx_2$$



The joint density  $f_{X_1, X_2}(x_1, x_2) = \frac{1}{4} \cdot \frac{1}{4}$  in the rectangular area so the double integral above is

$$\frac{1}{4} \cdot \frac{1}{4} \cdot |A|$$

where  $|A|$  is the area of the shaded III

and  $|A| = (5-1) \cdot 1 + (5-1) \cdot (6-4) \cdot \frac{1}{2} = 4 + 4 = 8$

$$\Rightarrow P(X_1 + 2X_2 \leq 13) = \frac{1}{4} \cdot 8 = \frac{1}{2}$$

Or: see that they  $\underline{\underline{=}}$

3 (cts.)

b) Alternatively see that the shaded part III has an area that is exactly half of the total area and  $f_{X_1, X_2}$  is constant in

~~so~~ The total area So

$$P((X_1, X_2) \in A) = \frac{1}{2}.$$

4.  $Q$  charge of ideal capacitor, hr density

$$f_Q(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$\tilde{Q}$  the total charge of capacitor stored in capacitor of type A.  $\tilde{Q}$  is a function of  $Q$  with

$$\tilde{Q} = \begin{cases} Q & \text{if } Q \leq 1 \\ 1 & \text{if } Q \geq 1 \end{cases}$$

a) Probability that capacitor of type A is fully charged is

$$\begin{aligned} P(\tilde{Q}=1) &= P(Q \geq 1) = \int f_Q(x) dx = \\ &= \int_1^{\infty} e^{-x} dx = -e^{-x} \Big|_1^{\infty} = e^{-1}. \end{aligned}$$

b) The density of  $\tilde{Q}$  is

$$f_{\tilde{Q}}(x) = \begin{cases} e^{-x} & , 0 \leq x \leq 1 \\ e^{-1} & , x = 1 \end{cases}$$

and is a density w.r.t mixed measure

$$\text{by Et.d.) } \mu(x) = \mu_0(x) \mathbf{1}_{\{x < 1\}} + \mu_1(x) \mathbf{1}_{\{x \geq 1\}}$$

where  $\mu_0$  length measure,  $\mu_1$  counting measure

$$\begin{aligned} c) E(\tilde{Q}) &= \int_{-\infty}^{\infty} x f_{\tilde{Q}}(x) d\mu(x) = \\ &= \int_0^1 x \cdot e^{-x} dx + \cancel{K.P.(\tilde{Q} = \emptyset)} 1 \cdot f_{\tilde{Q}}(1) \\ &= -x e^{-x} \Big|_0^1 + \int_0^1 e^{-x} dx + 1 \cdot e^{-1} \\ &= 0 - e^{-1} - e^{-x} \Big|_0^1 + 1 \cdot e^{-1} \\ &= \underline{1 - e^{-1}} \end{aligned}$$

5.  $N$  = number of particles that decay during  $[0, t]$   
 $N \in P_0(\Theta)$

$$P(N=0) = e^{-\Theta} \frac{\Theta^0}{0!} = e^{-\Theta}$$

It is known that  $P(N=0) = 0.1 = e^{-\Theta} \Rightarrow$

$$\Theta = -\log 0.1 \approx 2.2026$$

a)  $A = \{\text{success}\} = \{N \geq 2\}$

$$\begin{aligned} P(A) &= P(N \geq 2) = 1 - P(N \leq 1) \\ &= 1 - (P(N=0) + P(N=1)) \\ &= 1 - (0.1 + 0.1 \cdot \frac{(-\log 0.1)^1}{1!}) \\ &= 0.6697 \end{aligned}$$

b)  $n$  labs perform the experiment

$P(A) = 0.6697$  for each lab, and the experiments are independent. Let  $A_1, \dots, A_n$  be the events that success in experiment  $i = 1, \dots, n$ .  
 $P(\text{at least one is a success}) =$

$$\begin{aligned} P\left(\bigcup_{i=1}^n A_i\right) &= 1 - P\left(\bigcap_{i=1}^n A_i^c\right) = (\text{independent events}) \\ &= 1 - \prod_{i=1}^n P(A_i^c) = 1 - \prod_{i=1}^n (1 - P(A_i)) \\ &= 1 - (1 - 0.6697)^n \end{aligned}$$

5 (ctd.)

b) If  $n = 100$  we get

$$P\left(\bigcup_{i=1}^{100} A_i\right) = 1 - (1 - 0.6697)^{100} \approx 1.$$

Since

$$(1 - 0.6697)^{100} \approx 7.8 \cdot 10^{-49}$$

6.  $(\Omega, \mathcal{F}, P)$  a probability space

$\Xi_1, \Xi_2, \Xi_3, \dots$  i.i.d.  $E(\Xi_i) = 0$   $\text{Var}(\Xi_i) = 1$ .

$$U_i = \Xi_i + \Xi_{i+1}, \quad i = 1, 2, \dots$$

$$\begin{aligned} a) \quad E(U_i) &= E(\Xi_i + \Xi_{i+1}) = E(\Xi_i) + E(\Xi_{i+1}) \\ &= 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{Var}(U_i) &= \text{Var}(\Xi_i + \Xi_{i+1}) = (\text{indep.}) \\ &= \text{Var}(\Xi_i) + \text{Var}(\Xi_{i+1}) = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} \text{Cov}(U_i, U_{i+1}) &= \text{Cov}(\Xi_i + \Xi_{i+1}, \Xi_{i+1} + \Xi_{i+2}) \\ &= (\text{bilinear}) = \text{Cov}(\Xi_i, \Xi_{i+1}) + \text{Cov}(\Xi_i, \Xi_{i+2}) \\ &\quad + \text{Cov}(\Xi_{i+1}, \Xi_{i+1}) + \text{Cov}(\Xi_{i+1}, \Xi_{i+2}) \\ &= 0 + 0 + \text{Var}(\Xi_{i+1}) + 0 = 1. \end{aligned}$$

$$\begin{aligned} \text{Cov}(U_i, U_{i+k}) &= \text{Cov}(\Xi_i + \Xi_{i+1}, \Xi_{i+k} + \Xi_{i+k+1}) = (\text{bilinear}) \\ &= \text{Cov}(\Xi_i, \Xi_{i+k}) + \text{Cov}(\Xi_i, \Xi_{i+k+1}) \\ &\quad + \text{Cov}(\Xi_{i+1}, \Xi_{i+k}) + \text{Cov}(\Xi_{i+1}, \Xi_{i+k+1}) \\ &= 0 + 0 + 0 + 0 \quad (\cancel{\text{Bil}}) = 0. \end{aligned}$$

6 (ctd.)

b) We have that

$$E(U_1 + \dots + U_n) = E(U_1) + \dots + E(U_n) = 0 + \dots + 0$$

~~Theorem~~

Also

$$\begin{aligned} \text{Var}\left(\frac{U_1 + \dots + U_n}{n}\right) &= \frac{1}{n^2} \text{Var}(U_1 + \dots + U_n) \\ &= \frac{1}{n^2} \text{Cov}(U_1 + \dots + U_n, U_1 + \dots + U_n) \\ &= \frac{1}{n^2} \left( \text{Cov}(U_1, U_1) + \dots + \text{Cov}(U_n, U_n) + \right. \\ &\quad \left. \text{Cov}(U_1, U_2) + \text{Cov}(U_2, U_3) + \dots + \text{Cov}(U_{n-1}, U_n) \right. \\ &\quad \left. + \text{all rest are } 0 \right) \\ &= \frac{1}{n^2} \left( n \text{Var}(U_1) + (n-1) \text{Cov}(U_1, U_2) \right) \tag{1} \\ &= \frac{1}{n^2} (n \cdot 2 + (n-1) \cdot 1) = \frac{2n+n-1}{n^2} \rightarrow 0 \end{aligned}$$

c)  $n \rightarrow \infty$

And

$$\begin{aligned} \text{Var}\left(\frac{U_1 + \dots + U_n}{n}\right) &= E\left[\left(\frac{U_1 + \dots + U_n}{n} - E(\dots)\right)^2\right] \\ &= E\left(\left(\frac{U_1 + \dots + U_n}{n} - 0\right)^2\right) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , by (1). This shows that

6(ctd)

$$\frac{U_1 + \dots + U_n}{n} \xrightarrow{\mathbb{P}} 0$$

as  $n \rightarrow \infty$ .