

1. (Ω, \mathcal{F}, P) probability space

A, B events. $P(B \setminus A) = 0.2, P(A \cap B) = 0.1$



$A \cap B$

$$P(B \setminus A) = P(B) - P(A \cap B)$$

\Rightarrow

$$P(B) = 0.2 + 0.1 = 0.3$$

\Rightarrow

$$P(B^c) = 1 - P(B) = 1 - 0.3 = 0.7$$

2. V_1, V_2, V_3 independent r.v.'s, pmf

$$f_{V_k}(x) = p(1-p)^{x-1}$$

$x=1, 2, \dots$, $k=1, 2, 3$. The survival function is

$$S(x) = P(V_k > x)$$

For j an integer

$$P(V_k \geq j) = \sum_{x=j}^{\infty} f_{V_k}(x)$$

$$= \sum_{x=j}^{\infty} p(1-p)^{x-1} = p(1-p)^{j-1} \sum_{x=0}^{\infty} (1-p)^x$$

$$= p(1-p)^{j-1} \frac{1}{1-(1-p)} = (1-p)^{j-1}$$

Therefore

$$P(U \geq j) = P(\min(V_1, V_2, V_3) \geq j)$$

$$= P(V_1 \geq j, V_2 \geq j, V_3 \geq j)$$

indep. r.v.'s \rightarrow $= P(V_1 \geq j) \cdot P(V_2 \geq j) \cdot P(V_3 \geq j)$

$$= (1-p)^{j-1} (1-p)^{j-1} (1-p)^{j-1}$$

$$= ((1-p)^3)^{j-1}$$

set \tilde{p} be such that $1-\tilde{p} = (1-p)^3$, i.e. $\tilde{p} = 1-(1-p)^3$. Then U has pmf

$$f_U(x) = \tilde{p}(1-\tilde{p})^{x-1}$$

for $x=1, 2, \dots$. (Notice that $U = \min(V_1, V_2, V_3)$
can take the values $1, 2, 3, \dots$).

The expectation of U is

$$\begin{aligned} E(U) &= \sum_{x=1}^{\infty} x f_U(x) = \sum_{x=1}^{\infty} x \tilde{p}(1-\tilde{p})^{x-1} \\ &= \dots = \frac{1}{\tilde{p}} \end{aligned}$$

The variance of U is

$$\text{Var}(U) = E(U^2) - (E(U))^2$$

and

$$\begin{aligned} E(U^2) &= \sum_{x=1}^{\infty} x^2 f_U(x) = \sum_{x=1}^{\infty} x^2 \tilde{p}(1-\tilde{p})^{x-1} \\ &= \dots = \frac{1-\tilde{p}}{\tilde{p}^2} \end{aligned}$$

3. T_1, T_2 independent $\text{Exp}(1)$ distributed
r.v.'s

$$f_{T_i}(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for $i=1, 2$.

a) $V = e^{-T_1}$

V becomes a r.v. taking values in $(0, 1]$

$$P(V \leq v) = P(e^{-T_1} \leq v) = P(-T_1 \leq \log v)$$

$$= P(T_1 \geq -\log v) = 1 - P(T_1 < -\log v)$$

$$\xrightarrow[\text{r.v.}]{T_1 \text{ cont.}} 1 - P(T_1 \leq -\log v) \Rightarrow f_V(v) = -f_{T_1}(-\log v) \left(-\frac{1}{v} \right)$$

$$= \frac{1}{v} e^{-(-\log v)} = \frac{1}{v} \cdot v = 1$$

for $0 < v \leq 1$. Thus $V \in \text{Un}[0, 1]$.

b) $P\left(\frac{1}{4} \leq v + T_2 \leq \frac{3}{4}\right) = \iint f_{V, T_2}(v, t) dv dt$

$$= \int_{-\infty}^{\infty} \int_{\frac{1}{4}-v}^{\frac{3}{4}-v} f_V(v) f_{T_2}(t) dv dt$$

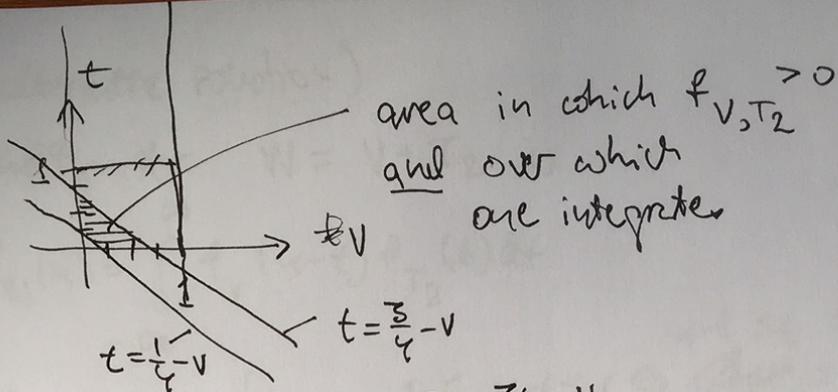
$(v, t): v+t \in \left[\frac{1}{4}, \frac{3}{4}\right]$

$$v = -\infty \quad t = \frac{1}{4} - v$$

=

4

6 ctd



$$\begin{aligned} &= \int_{v=0}^{1/4} \int_{t=\frac{1}{4}-v}^{3/4-v} e^{-t} dt dv + \int_{v=1/4}^{3/4} \int_{t=0}^{3/4-v} e^{-t} dt dv \\ &= \int_{v=0}^{1/4} \left[-e^{-t} \right]_{\frac{1}{4}-v}^{\frac{3}{4}-v} dv + \int_{v=1/4}^{3/4} \left[-e^{-t} \right]_{0}^{\frac{3}{4}-v} dv \\ &= \int_0^{1/4} -e^{-(\frac{3}{4}-v)} + e^{-(\frac{1}{4}-v)} dv + \int_{1/4}^{3/4} -e^{-(\frac{3}{4}-v)} + 1 dv \\ &= e^{-(\frac{3}{4}-v)} \Big|_0^{1/4} + e^{-(\frac{1}{4}-v)} \Big|_0^{1/4} + e^{-(\frac{3}{4}-v)} \Big|_{1/4}^{3/4} + \left(\frac{3}{4} - \frac{1}{4} \right) \\ &= e^{-(\frac{3}{4}-\frac{1}{4})} - e^{-\frac{3}{4}} + e^{-\frac{1}{4}} + 1 - e^{-(\frac{3}{4}-\frac{1}{4})} + \frac{1}{2} \\ &= \cancel{e^{-1/4} - \frac{1}{2}} = \cancel{0.2788} \\ &= e^{-1/4} - e^{-3/4} + \frac{1}{2} \end{aligned}$$

b) (ctd. alternative solution)

The density for $W = V + T_2$ is

$$f_W(u) = \int_{-\infty}^{\infty} f_V(u-t) f_{T_2}(t) dt$$

$$f_V(u-t) = \begin{cases} 1 & 0 \leq u-t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{T_2}(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow 0 \leq u \leq 1: \quad f_W(u) = \int_0^u 1 \cdot e^{-t} dt = -e^{-t} \Big|_0^u \\ = 1 - e^{-u}$$

$$u > 1: \quad f_W(u) = \int_{u-1}^u e^{-t} \cdot 1 dt = -e^{-t} \Big|_{u-1}^u \\ = e^{-(u-1)} - e^{-u}$$

$$\Rightarrow F_W\left(\frac{3}{4}\right) - F_W\left(\frac{1}{4}\right) = \int_{1/4}^{3/4} (1 - e^{-u}) du$$

$$= \frac{3}{4} - \frac{1}{4} + e^{-u} \Big|_{1/4}^{3/4} = \frac{1}{2} + e^{-3/4} - e^{-1/4}$$

5

4. The d.f. makes jumps at $x=1, 2, 3$ and is a diff function at $3 \leq x \leq 4$, and constant = 1 from 4. Thus there is a density w.r.t. μ the measure

$$\mu_0(x) \mathbb{1}_{\{x \leq 3\}} + \mu_1(x) \mathbb{1}_{\{x > 3\}}$$

μ_0 counting measure, μ_1 length measure.
Density

a)

$$f(x) = \begin{cases} \frac{1}{8} & x=1 \\ \frac{1}{8} & x=2 \\ \frac{1}{4} & x=3 \\ \frac{1}{2} & 3 \leq x \leq 4 \\ 0 & x \geq 4 \end{cases}$$

b)

$$\begin{aligned} E\left(\frac{1}{\bar{X}}\right) &= \sum_{x=1}^3 \frac{1}{x} f(x) + \int_3^4 \frac{1}{x} f(x) dx \\ &= 1 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{4} + \int_3^4 \frac{1}{x} \cdot \frac{1}{2} dx \\ &= \frac{13}{48} + \frac{1}{2} (\log 4 - \log 3) \end{aligned}$$

$$E(\mathbb{1}_{\{\bar{X}=3\}}) = P(\bar{X}=3) = \frac{1}{4}$$

5 a)

$$f_{U_i}(x) = \begin{cases} \frac{1}{2} & x=1 \\ \frac{1}{2} & x=2 \end{cases}$$

$$V_i = U_i^2. \quad E(V_i) = E(U_i^2) = \\ 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$$

By the LLN (in k^2 -sense)

$$\frac{1}{n}(V_1 + \dots + V_n) \xrightarrow{a.s.} E(V) = \frac{5}{2}$$

(Alt. Show the LLN....)

b) By the CLT

$V_1 + V_2 + \dots + V_n$
is for very large n , approximately
Gaussian.

$$E(V_1 + \dots + V_n) = n \cdot E(V_1) = n \cdot \frac{5}{2}$$

$$\text{Var}(V_1 + \dots + V_n) = \underset{\substack{\uparrow \\ \text{ind. r.v.'s}}}{n} \text{Var}(V_1) = n \cdot \frac{9}{4}$$

$$\begin{aligned} \text{Var}(V_1) &= E(V_1^2) - (E(V_1))^2 \\ &= E(U_1^4) - \left(\frac{5}{2}\right)^2 = 1^4 \cdot \frac{1}{2} + 2^4 \cdot \frac{1}{2} - \left(\frac{5}{2}\right)^2 \\ &= \frac{1}{2} + 8 - \left(\frac{5}{2}\right)^2 = \frac{9}{4} \end{aligned}$$

(7)

$$\frac{V_1 + \dots + V_n - n \cdot \frac{\sqrt{5}}{2}}{\sqrt{n \cdot \frac{9}{4}}} \stackrel{\sim}{\in} N(0,1)$$

so that

$$P(V_1 + \dots + V_n \leq 2530) \\ = P\left(\frac{V_1 + \dots + V_{1000} - 1000 \cdot \frac{\sqrt{5}}{2}}{\sqrt{1000 \cdot \frac{9}{4}}} \leq \frac{2530 - 1000 \cdot \frac{\sqrt{5}}{2}}{\sqrt{1000 \cdot \frac{9}{4}}}\right)$$

$$\stackrel{\sim}{=} \Phi\left(\frac{2530 - 1000 \cdot \frac{\sqrt{5}}{2}}{\sqrt{1000 \cdot \frac{9}{4}}}\right) = \Phi(0.6225)$$

$$\stackrel{\sim}{=} 0.7357$$

6. 10 processes started, attempted ...

X_1, \dots, X_{10} indicator variables

$$X_i = \begin{cases} 1 & \text{if process } i \text{ started} \\ 0 & \text{if not} \end{cases}$$

$P(X_i = 1) = p$ for $i = 1, \dots, 10$, X_1, \dots, X_{10} indep.

N_1, \dots, N_{10} indep $P_0(\theta)$ -distributed r.v.'s.

N_1, \dots, N_{10} ~~are~~ independent of the number of processes started.

a) $N = N_1 X_1 + N_2 X_2 + \dots + N_{10} X_{10}$

$$E(N) = E(N_1)E(X_1) + \dots + E(N_{10})E(X_{10})$$

$$= \theta \cdot p + \dots + \theta \cdot p = 10 \theta p$$

b) $\tilde{N} = N_1 + \dots + N_n$

$$P(\tilde{N} \geq 1) = 1 - P(\tilde{N} = 0)$$

$$P(\tilde{N} = 0) = P(\{N_1 = 0\} \cap \dots \cap \{N_n = 0\})$$

$$= \underset{\text{indep.}}{P(N_1 = 0) \dots P(N_n = 0)} = e^{-\theta} \frac{\theta^0}{0!} \dots e^{-\theta} \frac{\theta^0}{0!}$$

$$= (e^{-\theta})^n = e^{-\theta n}$$

Want $P(\tilde{N} \geq 1) = 1 - P(\tilde{N} = 0) \geq 0.9$

$$\Leftrightarrow P(\tilde{N} = 0) \leq 1 - 0.9 = 0.1$$

$$\Leftrightarrow e^{-\theta n} \leq 0.1 \Leftrightarrow -\theta n \leq \log 0.1$$

6b) contd.

$$\Leftrightarrow n \geq -\frac{\log 0.1}{\theta}$$

c) Since fewer attempted slots will be successful if $p < 1$, we will ~~not~~ need more attempts \hat{n} to obtain $P(\hat{N} \geq 1) \geq 0.9$, and therefore

the bound in b) is a lower bound for \hat{n} .