Lund Univerity
Mathematical Sciences
Mathematical Statistics 2016

Study questions for oral exam on first course in Mathematical Statistics, MASA01

1 Probability Theory

1. Define the concept of a σ-algebra, and give two examples of such. Define the outcome space, elementary events and events.

2. Define union, intersection and complements of events. What does it mean that two events are disjoint?

3. Explain how one can, for an arbitrary collection $\mathcal{A}$ of subsection of Ω, construct a σ-algebra from $\mathcal{A}$, and prove that the constructed thing is a σ-algebra. What is the (non-technical) reason for doing that?

4. Give the definition of a probability (so of a probability measure) P.

5. Derive from the definition of a probability measure the results: For $A, B \in \mathcal{F}$ with $\mathcal{F}$ a σ-algebra,

   (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

   (ii) If $A \subset B$ then $P(B) = P(A) + P(B \setminus A)$.

6. Let $\{A_i\}_{i=1}^{\infty}$ be a growing sequence $A_1 \subset A_2 \subset ...$ of events. Let $A = \cup_{i=1}^{\infty} A_i$. Show that $P(A) = \lim_{i\to\infty} P(A_i)$.

7. Define conditional probabilities. Define independent events.

8. Show Bayes theorem. Show the law of total probability.

9. Define the concepts of random variable (r.v.) and distribution function. Why is measurability important? Give an example of a function X that is not measurable.

10. Define the density of a r.v.. Give the definition of the three types of r.v.'s that you have studied. What is the density called in these cases?

11. Define the Riemann-Stiletjes integral $\int g(x)\, dF(x)$. Let X be a r.v. with distribution function F and g a function such that $E(g(X)) < \infty$. Give meaning to the expression $\int g(x)\, dF(x)$ and simplify it, for the three types r.v.'s that you have studied. Prove the simplified formulas from the definition of the Riemann-Stieltjes integral.

12. Define expectation, variance and standard deviation of a r.v..

13. Let $1\{A\}$ be the indicator function for the event A. Let $p = P(A)$. What is $E(1\{A\})$? What is $\text{Var}(1\{A\})$?

14. Define the Binomial distribution and describe how it arises. Derive the expressions for the probability mass function, from the description of how it arises. Derive the expressions for the expectation and the variance for this distribution.

15. Define the Poisson distribution and describe how it arises. Derive the expression for the variance.

16. Let X, Y be two r.v.'s. Give two equivalent definitions for them being independent, using d.f's or density functions. Show that the two definitions are equivalent.

17. Show that X, Y independent implies X, Y uncorrelated. Give an example for when the opposite implication does not hold.

18. $i)$ Suppose that (X, Y) is a continuous bivariate random vector. Define the conditional probability density function for X given Y. Define $E(X|Y)$. $ii)$ Suppose (X, Y) is discrete bivariate random vector. Define the conditional probability mass function for X given Y. Define $E(X|Y)$.

    Show that $E(X) = E(E(X|Y))$, for either the continuous or the discrete case.

19. Define the uniform $Un(0, \theta)$ distribution. Derive the expectation and the variance for it.

20. Define the exponential distribution $Exp(\theta)$. Derive the expectation and the variance for it.

21. Define the Normal distribution $N(\mu, \sigma^2)$. Show that if $X \in N(0, 1)$ then $aX + b \in N(b, a^2)$.

22. Derive the convolution theorem for $i)$ two discrete r.v.'s, $ii)$ two continuous r.v.. How does the formula simplify when the r.v.'s are independent?

23. Show that if $X.Y$ are independent and Normal, then $aX + bY$ is Normal.

24. Suppose that $X \in Bin(n_1, p)$, $Y \in Bin(n_2, p)$ and X, Y are independent. Derive the distribution of $X + Y$.

25. Define the four kinds of stochastic convergence that you have studied.

26. Give the statement of the Law of Large Numbers (LLN). Prove it. What is the use of the LLN?

27. Give the statement of the Central Limit Theorem (CLT). What is the use of the CLT?

2 Inference Theory

1. Let $x_1, \ldots, x_n$ be an i.i.d. sample from $X \sim F \in \{F_\theta : \theta \in \Theta\}$. What is the goal in inference theory? Define parametric, non-parametric and semi-parametric inference problems.

2. Define a statistic.

3. Define the empirical distribution F_n based on a sample. What is the distribution of $nF_n(x)$? Derive from this property $E(F_n(x))$ and $\mathrm{Var}(F_n(x))$.

4. Let $x_1, \ldots, x_n$ be i.i.d. observations of $X \sim F \in \{F_\theta : \theta \in \Theta\}$ and $\Theta \subset \mathbb{R}$. For $g : \mathbb{R} \to \mathbb{R}$, describe $E(g(X))$ as a functional of F. For which g does this definition make sense. Define the plug-in estimator of $E(g(X))$. Apply the definition on the estimation of $E(X)$ and of $Var(X)$.

5. What is an unbiased estimator? Define the mean square error (m.s.e.) of an estimator. Prove the partition of the m.s.e. into it's bias and variance part.

6. Define consistency of an estimator: in quadratic mean, almost surely and in probability. Show that consistency in quadratic mean implies consistency in probability.

7. Show that the plug-in estimator of a linear functional is consistent in quadratic mean.

8. Describe the Maximum Likelihood method for estimating a parameter θ.

9. Let $x_1, \ldots, x_n$ be a sample from $N(\mu, \sigma^2)$, with μ and σ^2 unknown. Derive the Maximum-Likelihood estimators of μ and σ^2.

10. Let x be a sample from $Bin(n, p)$ with p unknwon. Derive the Maximum Likelihood estimator of p.

11. Describe the least-squares method (LS) for estimating a parameter θ.

12. Derive the LS estimators for the parameters in question 9 and 10.

13. Describe the idea behind a confidence interval, with a certain confidence grade.

14. Describe the idea underlying statistical test. Define the concepts null- and alternative hypothesis, test statistics, critical region and significance level. Describe the p-value method. Define the power function for a given test.

15. Describe the method using a pivot function or pivot variable for constructing confidence intervals and tests.

16. Let $x_1, \ldots, x_n$ a sample from $N(\mu, \sigma^2)$, with μ unknwon, σ^2 known. Derive a 95% confidence interval μ. Derive a test on significance level 5% for some (choice yourself) hypotheses.

17. Derive an expression for the power function of the test in question 16.

18. Suppose that $x_1, \ldots, x_n$ is an i.i.d. sample from $N(\mu, \sigma^2)$, with both μ and σ^2 unknown. Derive a 95% confidence interval for μ. Derive a test on significance level 5% for some (chose yourself) hypotheses.

19. Let $x_1, \ldots, x_n$ be an i.i.d. sample from $N(\mu, \sigma^2)$, with both μ and σ^2 unknown. Derive a 95% confidence interval for σ^2. Derive a test on significance level 5% for some (chose yourself) hypotheses.

20. Describe the problem with simultaneous confidence intervals and repeated tests. Describe Bonferronis and Sidak correction methods.

21. Describe the the confidence method for doing tests. Describe how to do test-based confidence intervals.

22. How do you construct non-parametric one-sample tests with the use of the empirical distribution function? How do you construct confidence bands for F? How do you construct a k-sample test?

23. Suppose that $\theta = E(g(X))$ is a linear functional of F, and let $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^n g(X_i)$ be the plug-in estimator. Suppose that $\mathrm{Var}(g(X)) = \sigma^2 < \infty$ is known. What is the limit distribution of $\hat{\theta}_n$? Derive an approximate 95% confidence interval for $E(g(X))$. What happens when σ^2 is not known?

24. Suppose that $(x_{i1} \ldots, x_{ip}, y_i), i = 1, \ldots, n$ are observations in the regression problem

$$y_i = \alpha + \beta(x_{i1} - \bar{x}_{\cdot 1}) + \ldots \beta(x_{ip} - \bar{x}_{\cdot p}) + \epsilon_i,$$

with $\epsilon_i \in N(0, \sigma^2)$ i.i.d. r.v.. Write the regression problem on matrix form. Derive the least squares estimator $\hat{\beta}$ of the vector $\beta = (\beta_1, \ldots, \beta_p)$. Show that $\hat{\beta}$ is unbiased. Derive the covariance matrix of $\hat{\beta}$. What is the distribution of $\hat{\beta}$? How can you use this knowledge to make tests and confidence intervals for the β-parameters?

25. Let $(x_i, y_i), i = 1, \ldots, n$, be observations of the regression problem

$$y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i,$$

with ϵ_i idependent r.v.'s with $E(\epsilon_i) = 0, \mathrm{Var}(\epsilon_i) = \sigma^2$. What are the least squares estimators of α and β? Suppose that $\sigma^2 = \mathrm{Var}(\epsilon_1)$ is known and derive an approximate 95% confidence interval for α. Do the same thing for β.

26. Hur can you control if a linear model fits data (model validation)? How would you choose the order for a multivariate linear model?

27. Suppose that $x_1, \ldots, x_n$ are independent observations of a continuous r.v. X with unknown d.f. F and with $f = F'$ the p.d.f.. How can you estimate f? Why is this preferable to using the histogram?