Exercises for Chapter 2

1. Let Ω be the set of all individuals in Sweden (at a fixed but arbitrary time). Let A denote the set of all individuals in Sweden that are at most 25 years old, B the set of all individuals that are born in Skåne (Scania), and C the set of all male individuals. For simplicity assume that there are two genders: male and female. Describe with words the sets

   (a) A^c

   (b) $A \cap B^c$

   (c) $(A^c \cup B) \cap C^c$

   (d) $(A \cap B)^c \cup C$

2. Let A, B, C be sets. Simplify the expressions

$$(A \cup B^c) \cap (B^c \setminus A)$$
$$(A \cap C) \cup (A \cap C^c)$$

3. Let $A = [0, 1]$, and $B_n = [0, 1 + 1/n)$, for $n = 1, 2, \ldots$. Show that $A = \cap_{n=1}^{\infty} B_n$.

4. Let $A = [0, 1)$, and $C_n = [0, 1 - 1/n]$, for $n = 1, 2, \ldots$. Show that $A = \cup_{n=1}^{\infty} C_n$.

5. The Swedish parlament (Riksdagen) issued in 2015 a statement about the possible threat of a foreign state, Russia. The statement was that a military threat is "osannolik", which translates to *unlikely*. After a perceived hightened military presence near Swedens territory the so called "försvarsberedningen", which had as one of its objectives to assess possible military threats, issued a statement in 2018, that was also stated by the Swedish Supreme Commander ("överbefälhavare"), that a military attack "kan inte uteslutas", i.e. *can not be ruled out.* This was followed by the Swedish secretary of state and the Swedish prime minister arguing that the assessment of the government about the likelihood of an attack has not changed, and the formulation should not have been be sharpened. This in turn led to criticism from the parties in opposition, that seemed to be in favor of the statement from 2018. (Sources: dn.se, svd.se)

   Is the statement from 2018 a sharpening of the threat of conflict compared to the statement from 2015?

6. Let $\Omega = \{1, 2, \ldots, 10\}$ and $A = \{1, 2, 5\}$. Construct the σ-algebra $\sigma(A)$.

7. Let $\Omega = [0, 1]$ and $A_1 = (1/4, 3/4), A_2 = [0, 1/2]$. Construct the σ-algebra $\sigma(\{A_1, A_2\})$.

8. Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space.

   (a) Let A and B be events such that $P(A) = P(B) = 0$. Show that $P(A \cap B) = 0$.

   (b) Let A and B be events such that $P(A) = P(B) = 1$. Show that $P(A \cup B) = 0$.

9. (Example 2.32) Suppose that $\Omega = \{0, 1\}^\infty = \{0, 1\} \times \{0, 1\} \times \ldots$, which is a model of an infinite sequence of coin tosses, with 1 signifying "head" and 0 signifying "tail". Suppose that $\mathcal{F} = \mathcal{P}(\Omega)$.

   (a) Let

$$\begin{aligned} A_n &= \{0\} \times \ldots \times \{0\} \times \{1\} \times \Omega_{n+1} \times \Omega_{n+2} \times \ldots \\ &= (0, \ldots, 0, 1). \end{aligned}$$

   be the event that the first $n-1$ tosses result in "tail" and the n'th toss result in "head" (note the change of notation here relative to Example 2.32). Show that $A_n \downarrow$ to some $A \neq \emptyset$, and find A.

   (b) Let $B(k, l)$ be the event that there are exactly k "head" and l "tail" in an infinite sequence ω, where the "head" are put at arbitrary places *among the first* $n = k + l$ *positions* and the "tail" are put at (other) arbitrary positions *among the first* $n = k + l$ *positions*, and at all other positions one puts an $\{0, 1\}$ (i.e. whatever). Let p be a fixed given number in $(0, 1)$ and define the function $\tilde{P}_n$ on sets of type $B(k, l)$ as

$$\tilde{P}_n(B(k, l)) = \binom{k+l}{k} p^k (1-p)^l. \tag{1}$$

   If $B_n = \cup_{i=1}^\infty B(k_i, l_i)$ is a union of a countable number of *disjoint* sets of type $B(k, l)$, with $k + l = n$, define

$$\tilde{P}_n(B_n) = \sum_{i=1}^\infty \tilde{P}_n(B(k_i, l_i)).$$

   Show that the union, and the sum, is only finite. (Can you guess why we insist on using infinity nevertheless?)

(c) For any $C \in \mathcal{F}$, and any n, define $\tilde{C}_n$ as the restriction of C to the first n coordinates, so that $\tilde{C}_n$ is a union of sets of the form $B(k,l)$ for some k, l such that $k + l = n$. Define also the corresponding sets

$$C_n = \tilde{C}_n \times \Omega_{n+1} \times \Omega_{n+2} \times \ldots.$$

in $\mathcal{F}$. Suppose that $C, D \in \mathcal{F}$ are disjoint. Are the corresponding restricted sets $C_n, D_n \in \mathcal{F}$ necessarily disjoint?

(d) Show that $C_n \downarrow C$ as $n \to \infty$, for any $C \in \mathcal{F}$.

(e) Let $\tilde{P}_n$ be defined in (1). Define the functions P_n on the subcollection $\{C_n : C \in \mathcal{F}\}$ of $\mathcal{F}$ by

$$P_n(C_n) = \tilde{P}_n(\tilde{C}_n),$$

for all $n \geq 1$. Define the function P on $\mathcal{F}$, by

$$P(C) = \lim_{n \to \infty} P_n(C_n).$$

Show that in fact $P_n(C_n) \downarrow P(C)$.

(f) Show that P is a probability on $\mathcal{F}$.

(g) Calculate $P_n(A_n)$, where A_n is the event in (a).

(h) Show that $P(A) = 0$, where A is the event in (a).

Note that you have constructed an event A such that $A \neq \emptyset$ and $P(A) = 0$. Explain the statement in words.

10. Suppose that $\Omega = [0, 1]$, $\mathcal{F}$ is the σ-algebra generated by all open intervals in $[0, 1]$, and let μ be ordinary length measure (Borel measure) in Ω. Define the function

$$h : [0, 1] \to [0, 1]$$

by $h(x) = \mu([0, x])$. Show that $h(x)$ is right continuous.

11. (Example 2.32) Prove that the sequence of σ algebras $\{F_n\}_{n \geq 1}$, defined in Example 2.32, is a filtration. (Hint: Proof by induction.)

12. Let $\Omega = \{f : \mathbb{R} \to \mathbb{R}\}$ be a set of functions, and $t_0 \in \mathbb{R}$ a fixed point. Describe in words the set $A = A_{t_0}$, which is defined by

$$A = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} \left\{ f \in \Omega : |t - t_0| < \frac{1}{m} \Rightarrow |f(t) - f(t_0)| < \frac{1}{n} \right\}.$$