Fractional Factorial Design
(Chapter 6)

Agenda

Chapter 5:

- Short Recap of Two Level Factorial Design
  - The Replicated Design
  - Regression model

Chapter 6:

- Introduction to Fractional Factorial Design
- The One-Half Fraction of the 2^k Design
- Smaller Fractions - 2^{k-m} Fractional Factorial Designs

Recap - Example - 2^3 Factorial Design

Design factors	Low level $(-)$	High level $(+)$
A: Catalyzing substance	α	β
B: Concentration	5%	8%
C: Temperature	$80℃$	$90℃$

Experiment #	**Treatment Combination**	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	**Result, $\boldsymbol{y}$**
1	(1)	$-\ (\alpha)$	$-\ (5\%)$	$-\ (80℃)$	89 g
2	a	$+\ (\beta)$	$-\ (5\%)$	$-\ (80℃)$	84 g
3	b	$-\ (\alpha)$	$+\ (8\%)$	$-\ (80℃)$	131 g
4	ab	$+\ (\beta)$	$+\ (8\%)$	$-\ (80℃)$	130 g
5	c	$-\ (\alpha)$	$-\ (5\%)$	$+\ (90℃)$	124 g
6	ac	$+\ (\beta)$	$-\ (5\%)$	$+\ (90℃)$	121 g
7	bc	$-\ (\alpha)$	$+\ (8\%)$	$+\ (90℃)$	116 g
8	abc	$+\ (\beta)$	$+\ (8\%)$	$+\ (90℃)$	113 g

Short Recap – Two Level Factorial Design

- **Design matrix:**

Exp. No	A	B	C	AB	AC	BC	ABC	y
1	–	–	–	+	+	+	–	89
2	+	–	–	–	–	+	+	84
3	–	+	–	–	+	–	+	131
4	+	+	–	+	–	–	–	130
5	–	–	+	+	–	–	+	124
6	+	–	+	–	+	–	–	121
7	–	+	+	–	–	+	–	116
8	+	+	+	+	+	+	+	113
$\bar{E}_j$	-3	18	10	1	0	-26	-1	

Mean effect $= \frac{\sum_i y_i^+}{n/2} - \frac{\sum_i y_i^-}{n/2}, \quad n = 2^k$

- **The Replicated Case:** **Blackboard**

Regression Model for the Result y

- The analysis (from our previous example) tells us that the factors B and C together with the interaction BC are active
- The construction of the contrasts implies that the following regression model can be used to describe the experimental result y
  - $E_i =$ the average effect of the contrast L_i
  - $\bar{y} =$ average result over all experiments
  - $x_i =$ corresponding factor variable that can be high (+1) or low (-1), $i = A, B, C, AB, AC, BC, ABC$

$$\hat{y} = \bar{y} + \frac{E_B}{2} \cdot x_B + \frac{E_C}{2} \cdot x_C + \frac{E_{BC}}{2} \cdot x_B x_C$$

Regression Model for the Result y

- **Notice that:** The regression coefficient is measuring the effect when x_B is increased by *one* unit. However, in factorial experiments x_B is increased from -1 to $+1$, i.e., two units. Therefore, we must divide E_B by 2 in the regression model

- **Ex:** $\boldsymbol{x_B = x_C = +1 \Longrightarrow x_B x_C = +1}$
  - $\bar{y} = 113.5$
  - $\hat{y} = 113.5 + \frac{18}{2} \cdot (+1) + \frac{10}{2} \cdot (+1) + \frac{-2}{2} \cdot (+1) = 114.5$

Fractional (or Reduced) Factorial Design (Chapter 6)

- Requires fewer experiments but (interaction) effects are confounded $\Rightarrow$ cannot analyze/distinguish all interactions
- May save time and money if it is known that certain interactions between factors are of no importance!
- Often used as a screening tool to figure out which factors should be more carefully studied in a complete factorial design of experiments
- Useful in quality improvement because often the results are mainly influenced by a small number of factors (Pareto's law – the vital few and the trivial many)

Fractional Factorial Design

- Suppose there are k factors (A,B,...,J,K) in an experiment. All possible factorial effects include

  effects of order 1: A, B, ..., K (main effects)

  effects of order 2: AB, AC, ...,JK (2-factor interactions)

  effects of order 3: … Etc.

- Hierarchical Ordering principle
  - Lower order effects are more likely to be important than higher order effects
  - Effects of the same order are equally likely to be important
- Pareto principle
  - The number of important effects in a factorial experiment is small

Example

Suppose you were designing a new forklift:

- Want to consider the following nine factors each with 2 levels
- 1. Engine Size; 2. Lifting chain; 3. Fork; 4. Weight; 5. Automatic vs Manual; 6. Shape; 7. Tires; 8. Suspension; 9. Battery;
- Only have resources for conducting $2^6 = 64$ runs
  - If you drop three factors for a 2^6 full factorial design, those factor and their interactions with other factors cannot be investigated
  - Want investigate all nine factors in the experiment
  - A fraction of 2^9 full factorial design will be used
  - Confounding (aliasing) will happen because using a subset
- **How to choose (or construct) the fraction?**

The One-Half Fraction of the 2^k Design

- Suppose there are four factors in an experiment (A, B, C and D), each of 2 levels.
- Suppose the available resource is enough for conducting 8 runs
- 2^4 full factorial design consists of all the 16 level combinations of the four factors
- We need to choose half of them
- The chosen half is called a 2^{4-1} fractional factorial design

All Combinations in a Full 2^4 Design

A	B	C	D
-	-	-	-
+	-	-	-
-	+	-	-
+	+	-	-
-	-	+	-
+	-	+	-
-	+	+	-
+	+	+	-
-	-	-	+
+	-	-	+
-	+	-	+
+	+	-	+
-	-	+	+
+	-	+	+
-	+	+	+
+	+	+	+

Design matrix

Construct 2^{4-1} designs via "confounding" (aliasing)

- Select 3 factors (e.g. A, B, C) to form a 2^3 full factorial design
- Confound (alias) D with a high order interaction of A, B and C. For example, D = ABC

A	B	C	AB	AC	BC	D= ABC
-	-	-	+	+	+	-
+	-	-	-	-	+	+
-	+	-	-	+	-	+
+	+	-	+	-	-	-
-	-	+	+	-	-	+
+	-	+	-	+	-	-
-	+	+	-	-	+	-
+	+	+	+	+	+	+

Defining Relation

- The chosen level combinations form a half of the 2^4 design
- The product of columns A, B, C and D equals 1, i.e.,

$$I = ABCD$$

which is called the **defining relation**, and ABCD is called a **defining word** (contrast)

- $I = ABCD$ is determined by the relation $D = ABC$

Defining Relation

- Multiply the defined equality with the aliased factor:

$$I = D \cdot D = (ABC) \cdot D = ABCD$$

- That is:

$$\begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ +1 \end{bmatrix} = \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \\ +1 \\ +1 \\ +1 \\ +1 \end{bmatrix} = I$$

Design 1: Alias Structure for 2^{4-1} with $\underline{I = ABCD}$

- $I = ABCD$
- $A = A \cdot I = A \cdot ABCD = BCD$

By the same procedure:

- $B = ACD$
- $C = ABD$
- $D = ABC$
- $AB = AB \cdot I = AB \cdot ABCD = CD$
- $AC = BD$
- $AD = BC$

Design 2: Alias Structure for 2^{4-1} with $\underline{I = ABD}$

The defining relation $I = ABD$ generates another 2^{4-1} fractional factorial design, denoted by *design 2*. Its alias structure is given by:

- $I = ABD$
- $A = BD$
- $B = AD$
- $C = ABCD$
- $D = AB$
- $ABC = CD$
- $ACD = CD$
- $BCD = AC$

- Recall that *design 1* is defined by $I = ABCD$
- Comparing *design 1* and *design 2*, **which one we should choose?**

Word Length and Resolution

- **Length** of a defining word is defined to be the number of the involved factors
- **Resolution** of a fractional factorial design is defined to be the minimum length of the defining words, usually denoted by Roman numbers, III, IV, V,...

Resolution

- Design 1: $I = ABCD$ is a resolution IV design denoted by 2_{IV}^{4-1}
- Design 2: $I = ABD$ is a resolution III design denoted by 2_{III}^{4-1}
- A design is of resolution R if no p-factor effect is aliased with another effect containing less than $R - p$ factors
  - Design 1: main effects are not aliased with other main effects and 2-factor interactions
  - Design 2: main effects are not aliased with main effects
- Design 1 is better than Design 2, because Design 1 has higher resolution than Design 2

Our Example with Design 1

Contrasts								
	BCD	***ACD***	***ABD***	***CD***	***BD***	***AD***	***D***	
	A	***B***	***C***	***AB***	***AC***	***BC***	***ABC***	
Exp. No								**y**
1	–	–	–	+	+	+	–	89
2	+	–	–	–	–	+	+	84
3	–	+	–	–	+	–	+	131
4	+	+	–	+	–	–	–	130
5	–	–	+	+	–	–	+	124
6	+	–	+	–	+	–	–	121
7	–	+	+	–	–	+	–	116
8	+	+	+	+	+	+	+	113
$\bar{E}_j$	-3	18	10	1	0	-26	-1	

From the previous analysis statistically significant contrasts are:

- (B+ACD) with average effect = 18
- (BC+AD) with average effect = -26
- Potentially also *(C+ABD) with average effect = 10*

Confounding Patterns for Reduced Factorial Designs with Multiple Generating Relations

- Notation 2_W^{k-m}

  2 = number of factor levels
  k = number of factors
  m = number of generating relations
  W = shortest "word" in the defining relation

- Use a confounding table for reduced 2-factorial design of experiments for example available in Table 6.22 (course book)
- **Example:** Assume that we want to investigate the impact of 7 different factors with as few experiments as possible
- Need a design matrix with at least 7 contrasts one for each main factor which implies that at least 8 experiments must be carried out

  ⇒ Requires a 2^{7-4} reduced factorial design

- **The confounding table renders the design:**

$$2_{III}^{7-4} \quad \begin{array}{l} \pm 4 = 12 \\ \pm 5 = 13 \\ \pm 6 = 23 \\ \pm 7 = 123 \end{array}$$

Reduced Factorial Design Matrix

- Generating relations from confounding table (Table 6.22)

$$D = AB, \qquad E = AC, \qquad F = BC, \qquad G = ABC$$

Contrasts								
				D	**E**	**F**	**G**	
Exp. No	***A***	***B***	***C***	***AB***	***AC***	***BC***	***ABC***	**y**
1	−	−	−	+	+	+	−	
2	+	−	−	−	−	+	+	
3	−	+	−	−	+	−	+	
4	+	+	−	+	−	−	−	
5	−	−	+	+	−	−	+	
6	+	−	+	−	+	−	−	
7	−	+	+	−	−	+	−	
8	+	+	+	+	+	+	+	

Components of the Defining Relation

- From the generating relations we can directly obtain the initial components of the defining relation (sometimes referred to as defined equalities)
- $D = AB \Rightarrow I = ABD$
- $E = AC \Rightarrow I = ACE$
- $F = BC \Rightarrow I = BCF$
- $G = ABC \Rightarrow I = ABCG$

$$\Rightarrow I = ABD = ACE = BCF = ABCG$$

- The complete Defining Relation DR is obtained by multiplying the initial components of the defining relation with each other in all possible ways:
  - **(1) Pairwise:** No of combinations $= \binom{4}{2} = 6$
  - **(2) Three & Three:** No of combinations $= \binom{4}{3} = 4$
  - **(3) Four & Four:** No of combinations $= \binom{4}{4} = 1$

Components of the Defining Relation

- **(Multiplication Pairwise)**
  - (ABD)·(ACE) = BCDE
  - (ABD) (BCF) = ACDF
  - (ABD) (ABCG) = CDG
  - (ACE) (BCF) = ABEF
  - (ACE) (ABCG) = BEG
  - (BCF) (ABCG) = AFG
- **(Multiplication Three&Three)**
  - (ABD)·(ACE) (BCF) = DEF
  - (ABD)·(ACE) (ABCG) = ADEG
  - (ABD)·(BCF) (ABCG) = BDFG
  - (ACE)·(BCF) (ABCG) = CEFG
- **(Four&Four)**
  - (ABD)·(ACE) (BCF) (ABCG) = ABCDEFG

- The complete Defining set of relations, DR, is then:

DR: I = ABD= ACE= BCF= ABCG= BCDE = ACDF=
= CDG = ABEF = BEG = AFG = DEF = ADEG =
= BDFG = CEFG = ABCDEFG

Confounded interactions

- The interactions confounded on *A* can now be obtained by multiplying the complete Defining Relation, DR, with *A*:

  A·DR gives:

  A = BD= CE= ABCF= BCG= ABCDE = CDF=

  = ACDG = BEF = ABEG = FG = ADEF = DEG =

  = ABDFG = ACEFG = BCDEFG

- The interactions confounded on the other main factors and thereby the confounding patterns for all contrasts can be determined in the same manner.

  $X_j \cdot DR \Rightarrow$ all factor interactions confounded on the main factor X_j

- By studying the Defining Relation, DR we can conclude:
  - The shortest word length is 3 (i.e. at least three way interactions). We say the experimental design has a resolution of 3
  - Main factors will be confounded on no less than 2-way interactions.

  $X_m \cdot DR \Rightarrow$ "smallest" interaction is between two factors $X_i X_j$