

Lecture 2

Mathematical prerequisites

X r.v. $F_X(x) = P(X \leq x)$

discrete r.v. pmf f_X , P_X
 $P_X(w) = P(X = w)$

continuous r.v. pdf f_X , P_X

$$\int_A f_X(u) du = \cancel{P(\cancel{X})} \\ = P(X \in A)$$

X r.v. with density, f_X , P_X , w.r.t. a measure μ , means that

$$F_X(x) = \int_{-\infty}^x f_X(u) d\mu(u) \quad (1)$$

(i) discrete X , μ = counting measure

$$(1) \Leftrightarrow F_X(x) = \int_{-\infty}^x f_X(u) d\mu(u)$$

$$= \sum_{k \leq x} f_X(k)$$

(ii) continuous r.v. \bar{X} , μ = length measure

$$(ii) \Leftrightarrow F_{\bar{X}}(x) = \int_{-\infty}^x f_{\bar{X}}(u) du$$

Group families of distributions

One of two general classes of families of distributions that we will look at.

Start with a r.v. U (suppose that U is a cont. r.v.). Then f_U its d.f. F_U and density f_U (probability density function). Transform U to the r.v.

$$\bar{X} = a + bU \quad , \quad a \in \mathbb{R}, \quad b \in \mathbb{R}^+, \quad b > 0$$

Then

$$\begin{aligned} F_{\bar{X}}(x) &= P(\bar{X} \leq x) = P(a + bU \leq x) \\ &= P(bU \leq x - a) \\ &= P(U \leq \frac{x-a}{b}) \\ &= F_U\left(\frac{x-a}{b}\right) \end{aligned}$$

②

We can start with the d.f. $F = F_0$
 and we can form the family of distributions

$$(2) \quad \left\{ F_{a,b}(x) = F\left(\frac{x-a}{b}\right) : \begin{array}{l} a \in \mathbb{R}^+ \\ b > 0 \end{array} \right\}$$

Recall that for our inference problem
 we study parametric families

$$\{F_\theta : \theta \in \Theta\}$$

↑ ↗
 parameter parameter space

$$\text{Now: } \Theta = (a, b), \quad \mathcal{R} \ni \theta = \mathbb{R} \times \mathbb{R}_+$$

We can view as a family generated
 by the transformation $g_{a,b}$

$$g_{a,b}(x) = a + bx \quad (3)$$

where $a \in \mathbb{R}, b > 0$.

Let \mathcal{G} be the set of transformations

$$\mathcal{G} = \{g_{a,b} : a \in \mathbb{R}, b \in \mathbb{R}_+\}$$

③

Then \mathcal{G} is a group under group operation composition.

$$g_1, g_2 \in \mathcal{G}$$

$$g_1 \cdot g_2 = g_1 \circ g_2 \quad (g_1(g_2(x)) \\ = g_1 \circ g_2(x))$$

(i) $g_1, g_2 \in \mathcal{G}$ then $g_1 \circ g_2 \in \mathcal{G}$, since

$$\begin{aligned} g_1(g_2(x)) &= g_1(a_2 + b_2 x) = \\ &= a_1 + b_1(a_2 + b_2 x) = \underbrace{a_1 + b_1 a_2}_{\mathcal{G}} + \underbrace{b_1 b_2 x}_{\mathcal{G}} \end{aligned}$$

(ii) On $g \in \mathcal{G}$ then (there is an) $g^{-1} \in \mathcal{G}$, since

$$y = g(x) = a + bx \Leftrightarrow$$

$$x = \frac{y-a}{b} = -\frac{a}{b} + \frac{1}{b}y$$

$\underbrace{a}_{\mathcal{G}}$ $\underbrace{b}_{\mathcal{G}}$

if associative under operation

Def of a group if \exists identity,

\exists inverse (closed under these operations)

Problem 4.4.

4

Examples

1) Normal, Gaussian $N(a, b^2)$

$$f(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-(x-a)^2/2b^2}$$

$a \in \mathbb{R}, b > 0$.
 expectation
 standard deviation

$\bar{X} \in N(a_1, b_1^2)$ Then

$$Y = a_2 + b_2 \bar{X} \in N(a_2 + b_2 a_1, b_2^2 b_1^2)$$

2) Exponential distribution $E(a, b)$

$$f_{\bar{X}}(x) = \frac{1}{b} e^{-(x-a)/b} \cdot 1\{x \geq a\}$$

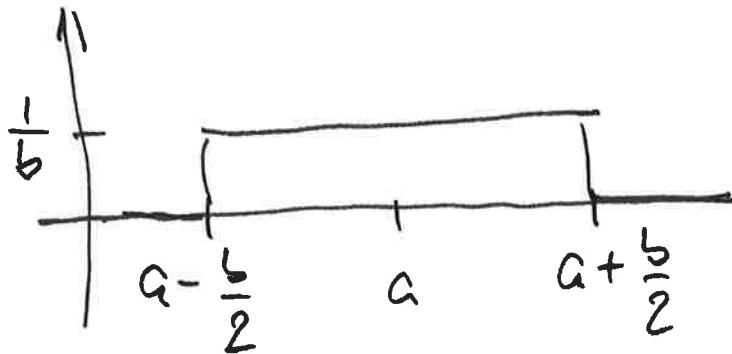
Group family. Check!

3) Uniform distribution $Un(a - \frac{b}{2}, a + \frac{b}{2})$

$$f(x) = \frac{1}{b} 1\left\{a - \frac{b}{2} \leq x \leq a + \frac{b}{2}\right\}$$

$$= \begin{cases} \frac{1}{b} & a - \frac{b}{2} \leq x \leq a + \frac{b}{2} \\ 0 & \text{otherwise} \end{cases}$$

(5)



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Special cases we will look at are

$$\{ F(x-a) : a \in \mathbb{R} \} \text{ location family}$$

$$\{ F\left(\frac{x}{b}\right) : b > 0 \} \text{ scale family}$$

$$\{ F\left(\frac{x-a}{b}\right) : a \in \mathbb{R}, b > 0 \} \text{ location-scale family}$$

a location

b scale

Ex: $U = (U_1, U_2, \dots, U_n)$ random vector
then

$$U+a = (U_1+a, \dots, U_n+a) \quad a \in \mathbb{R}$$

$$bU = (bU_1, \dots, bU_n) \quad b > 0$$

$$a+bU = (a+bU_1, \dots, a+bU_n) \quad a \in \mathbb{R} \\ b > 0$$

⑥ resp. location, scale and location-scale families

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Ex: (multivariate Gaussian)

Let $U = (U_1, \dots, U_p)$ i.i.d. $N(0, 1)$

Define the new random vector

vector

$$\bar{X} = a + B U$$

where

$$\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} + B \begin{pmatrix} U_1 \\ \vdots \\ U_p \end{pmatrix}$$

B invertible $p \times p$ -matrix.

We see that

$$f_{\bar{X}(U)} = \frac{1}{(2\pi)^{p/2}} e^{-\frac{\sum u_i^2}{2}}$$

$$f_U(u) = \frac{1}{(2\pi)^{p/2}} e^{-\frac{\sum u_i^2}{2}}$$

By the change of variable formula
we get that

(7)

$$f_{\alpha, B}(x) = f_{\bar{x}}(x) = \frac{1}{(2\pi)^{p/2} |B|^{p/2}} e^{-(x-\alpha)^T \Sigma^{-1} (x-\alpha)/2} \quad (4)$$

where

$$E(\bar{x}) = \alpha$$

$$\begin{aligned} \text{cov}(\bar{x}) &= \text{var}(\bar{x}) = E(\bar{x} \bar{x}^T - E(\bar{x}) E(\bar{x})^T) \\ &= B B^T = : \Sigma \end{aligned}$$

Then

$\{f_{\alpha, B}(x) : \alpha \in \mathbb{R}^p, B \text{ invertible } p \times p \text{ matrix}\}$

is the set of all non-degenerate Gaussian vectors of length p .

(Problems 4.7 & 4.8).

Read about linear models.

Ex: Suppose U r.v. $N(0,1)$.

$G = \{g : \mathbb{R} \rightarrow \mathbb{R} ; \lim_{u \rightarrow -\infty} g(u) = -\infty,$

$\lim_{u \rightarrow \infty} g(u) = \infty, g$ strictly increasing } g cont.

This is a group.

Define $\bar{X} = g(U)$. Then we get
the family of all cont r.v. with
support in \mathbb{R} . $F = F_{N(0,1)}$ and

F_g is the distribution function of
 $\bar{X} = g(U)$ i.e. $F_{\bar{X}} = F \circ g^{-1}$, the
 $\{F_g : g \in G\}$

the family of all cont. distributions
with support \mathbb{R} in $(-\infty, \infty)$.

⑨

Read the example about symmetric distributions.

Note : A group family of distributions does not depend on the ~~starting~~ starting distribution.

Ex: \cup the cont. r.v. which is $U \in [0,1]$

Let

$h = \{g : \text{cont., strictly increasing}$
 $\text{on } [0,1], g(0) = 0,$
 $g(1) = 1\}$

Let

$$X = a + bg(U), \quad a \in \mathbb{R}, \quad b > 0$$

Then

$$F_X = F_{a,b,g}$$

is the d.f. of a cont. r.v. with support on $(a, a+b)$. Thus

$$\{F_{a,b,g} : a \in \mathbb{R}, b > 0, g \in h\}$$

is the set of all cont. distributions with support in $(a, a+b)$.

Notes

This course is about

parametric problems
finite sample properties.

Inference problem:

x_1, \dots, x_n i.i.d. r.v. $F_0 \in \{F_\theta : \theta \in \mathcal{L}\}$
 \rightarrow parameter space $= \mathcal{L}$

$\dim(\mathcal{L}) < \infty$ parametric problem

$\dim(\mathcal{L}) = \infty$ nonparametric problem

Ex: nonparametric

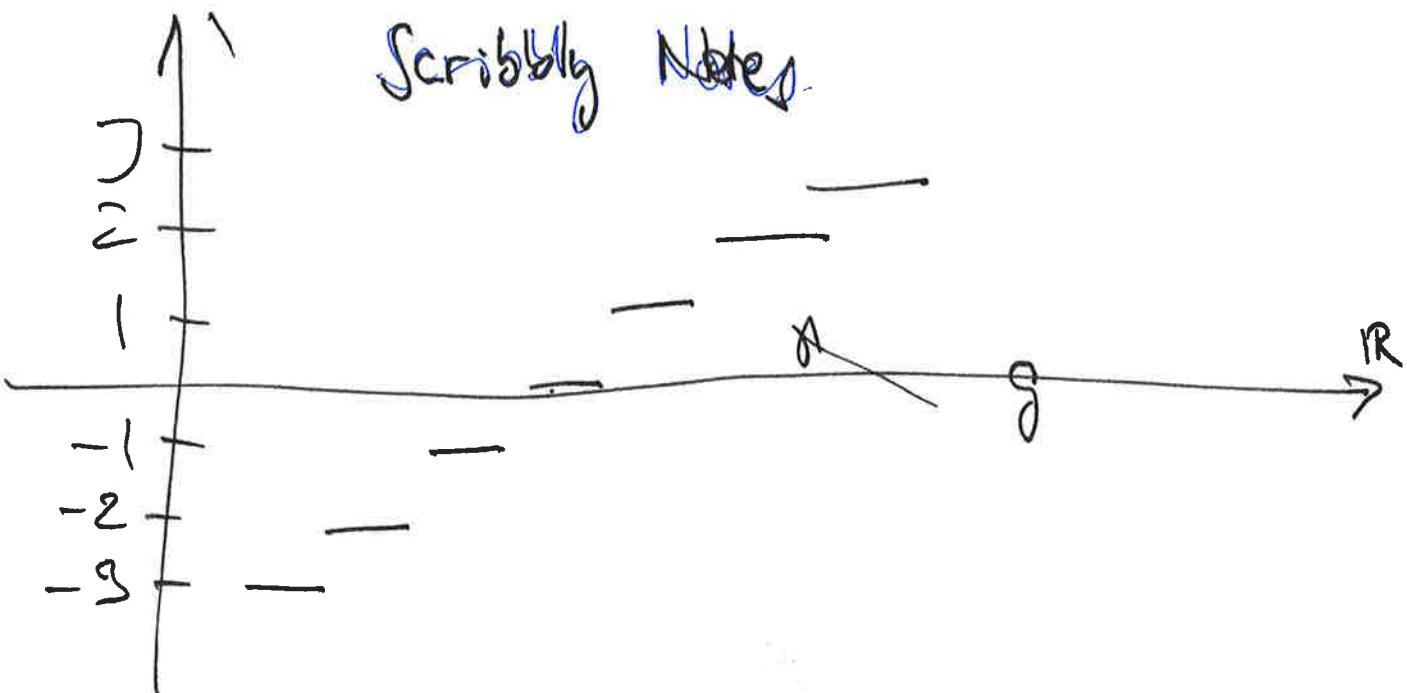
$\mathcal{L} = \{f \in C^2 : f = 0, f \geq 0, \int f d\mu = 1\}$

$\mathcal{F}_0 = \{ \text{density } f \text{ s.t. } f \in \mathcal{L} \}$

$\mathcal{F}_1 = \{ \text{density } f \neq 0 \text{ on } \mathbb{R}_+ \}$

Ex: parametric

$\mathcal{F}_2 = \{ f_{N(\mu, \sigma^2)} : \theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+ \}$



$$X = g(U)$$

$$\text{supp } f = \overline{\{x : f(x) \neq 0\}}$$

sup_{po} \rightarrow support of the distribution
 of X
 of X cont r.v.

$$\text{supp } f_X$$

$$\overline{(0, 1)} = [0, 1]$$