

The Neyman-Pearson Lemma

Lecture 19

Suppose H and K consist of only one probability distribution each.

$$K = \{P_1\}$$

$$H = \{P_0\}$$

Suppose the r.v. is discrete so $\mathcal{X} = \{x_1, x_2, \dots\}$ is the possible ~~wrong~~ values it can take.

Introduce the probability mass functions $P_0(x), P_1(x)$

$$P_0(\bar{X}=x) =: P_0(x)$$

$$P_1(\bar{X}=x) =: P_1(x).$$

For a nonrandomized test, we need a critical region S , such that if $\alpha \in (0, 1)$ is the significance level, we can define the optimal level α test, as the test (critical region S) that maximizes

$$\sum_{x \in S} P_1(x) \quad (1)$$

①

under the constraint

$$\sum_{x \in S} P_0(x) \leq \alpha. \quad (2)$$

Let us look at choosing the points x that should go into S . When doing that we want to make the contribution to (1) as large as possible while making a contribution to (2) as small as possible. We should therefore pick an x which makes

$$r(x) = \frac{P_1(x)}{P_0(x)}$$

as large as possible. Consider the x according the values of $r(x)$, from largest to smallest, keep on adding points x into S , until we can't any more due to (2).

②

The solution \mathcal{J} is the set of points x such that $r(x) > c$, with c chosen so that

$$\begin{aligned} P_0(\bar{X} \in \mathcal{J}) &= \sum_{x \in \mathcal{J}} P_0(x) \\ &= \sum_{x: r(x) > c} P_0(x) = \alpha \end{aligned} \quad (3)$$

Problem here, since \bar{X} discrete it might not be possible to get exactly $= \alpha$ in (3). The elegant way to solve this is using randomized test.

Theorem (Neyman - Pearson)

Let P_0, P_1 be probabilities with densities p_0, p_1 w.r.t. measure μ . We want to test

$$\begin{array}{ll} H: & p_0 \\ K: & p_1 \end{array}$$

(3)

at level α . Then

(i) (existence) There is a test φ and a constant K' such that

$$(9) \quad E_0 \varphi(\bar{x}) = \alpha$$

and

$$(10) \quad \varphi(x) = \begin{cases} 1 & \text{if } p_1(x) > K' p_0(x) \\ 0 & \text{if } p_1(x) < K' p_0(x) \end{cases}$$

(ii) (sufficiency) If a test φ satisfies (9) and (10) for some K' , then it is most powerful at level α .

(iii) (necessity) If φ is most powerful at level α , then (10) holds for some K' , μ -a.s. It satisfies also (9), unless there is a test of size strictly smaller than α with power 1.

Proof Suppose $\alpha \in (0, 1)$.

(i) Define

$$\begin{aligned}\alpha(c) &= P_0 \left(p_1(\bar{x}) > c p_0(\bar{x}) \right) \\ &= \int [1 \{ p_1(x) > c p_0(x) \}] dP_0(x) \\ &= \int [1 \{ p_1(x) > c p_0(x) \}] p_0(x) d\mu(x)\end{aligned}$$

Points where $p_0(x) = 0$ give zero contribution to the integral, we only consider points where $p_0(x) > 0$. Therefore

$$\alpha(c) = P_0 \left(\frac{p_1(\bar{x})}{p_0(\bar{x})} > c \right)$$

so that $1 - \alpha(c)$ is a distribution function.

Therefore $\alpha(\bar{x})$ is decreasing and right-continuous. Furthermore

$$P_0 \left(\frac{p_1(\bar{x})}{p_0(\bar{x})} = c \right) = \alpha(c-) - \alpha(c)$$

$$\alpha(-\infty) = 1$$

⑤

Now take c_0 such that

$$\alpha(c_0) \leq \alpha \leq \alpha(c_0^-)$$

and define

$$\begin{aligned}\varphi(x) = & 1 \{ p_1(x) > c_0 p_0(x) \} \\ & + \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} 1 \{ p_1(x) = c_0 p_0(x) \}\end{aligned}$$

Now suppose first that $\alpha(c_0^-) < \alpha(c_0)$.

The ~~size~~^{level} of the test $p(\cdot)$

$$E_0 \varphi(x) = P_0(p_1(x) > c_0 p_0(x))$$

$$+ \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} P_0(p_1(x) = c_0 p_0(x))$$

$$= \alpha(c_0) + \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} \cdot (\alpha(c_0^-) - \alpha(c_0))$$

$$= \alpha$$



⑥

We can choose $k' = c_0$ to get (i).

If instead $\alpha(c_0-) = \alpha(c_0)$ then the second term does not make sense,
but then also $E_0 1\{p_1(\bar{x}) = c_0 p_0(\bar{x})\} = 0$
so we can define $\alpha \cdot 0 = 0$.
So again we get (i).

(ii) Suppose φ satisfies (9) and (10).

Let φ^* another test with $E_0 \varphi^*(\bar{x}) \leq \alpha$.

Let

$$S^+ = \{x : \varphi(x) > \varphi^*(x)\}$$

$$S^- = \{x : \varphi(x) < \varphi^*(x)\}$$

:

On S^+ , since $\varphi^* \geq 0$, we must have
 $\varphi(x) > 0$, and $p_1(x) \geq k' p_0(x)$.

Similarly when $x \in S^-$ we have $\varphi(x) < 1$
and $p_1(x) \leq k' p_0(x)$. Therefore

⑦

$$\int (\varphi - \varphi^*) (p_1 - k' p_0) d\mu$$

$$= \int_{S^+ \cup S^-} (\varphi - \varphi^*) (p_1 - k' p_0) d\mu \geq 0 \quad (*)$$

Thⁿ) implies

$$\begin{aligned} \bar{E}_1 \varphi(\mathbb{X}) - \bar{E}_1 \varphi^*(\mathbb{X}) &= \frac{\int \varphi(x) p_1(x) d\mu(x)}{- \int \varphi^*(x) p_1(x) d\mu(x)} \\ &= \int (\varphi(x) - \varphi^*(x)) p_1(x) d\mu(x) \\ (*) &\geq k' \int (\varphi(x) - \varphi^*(x)) p_0(x) d\mu(x) \\ &= k' [\bar{E}_0(\varphi(\mathbb{X})) - \bar{E}_0(\varphi^*(\mathbb{X}))] \\ &\geq 0 \end{aligned}$$

so that $\bar{E}_1 \varphi(\mathbb{X}) \geq \bar{E}_1 \varphi^*(\mathbb{X})$

and φ most powerful.

(R)

(iii) Suppose φ^* most powerful and φ another test, the test given by (9) and (10). Define S^+, S^- as above and

$$S = (S^+ \cup S^-) \cap \{x : p_1(x) \neq k' p_0(x)\}$$

Then on S we get, reasoning as above

$$(\varphi - \varphi^*)(p_1 - k' p_0) > 0 \quad \text{note by } \textcircled{P}$$

Suppose $\mu(S) > 0$, then

$$\int_{S^+ \cup S^-} (\varphi - \varphi^*)(p_1 - k' p_0) d\mu$$

$$= \int_S (\varphi - \varphi^*)(p_1 - k' p_0) d\mu > 0$$

but this means that, reasoning as in (ii)

$$E_1 \varphi(X) - E_1 \varphi^*(X) > 0$$

~~Q~~ which means that φ more powerful than

p^* ; but φ^* was most powerful, therefore $M(S) = \cup$, so that $\varphi = p^*$ a.e. a.s., and φ^* satisfies ~~(H), (T)~~. (H).

We need check (S). If φ^* is of level strictly smaller than α then we can add ^{and} points additional points to the rejection region until either $E_0\varphi^*(X) = \alpha$ or $E_1\varphi^*(X) = 1$.

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Ex. $X \in N(\xi, \sigma^2)$ σ^2 known.

Hypotheses

$$H: \xi = 0$$

$$K: \xi = \xi_1$$

for fixed ξ_1 .

Then

$$\frac{p_1(x)}{p_0(x)} = \frac{e^{-(x-\xi_1)^2/2\sigma^2}}{e^{-x^2/2\sigma^2}}$$
$$= e^{\xi_1 x / \sigma^2 - \xi_1^2 / 2\sigma^2}$$

$\xi_1 > 0$, and exp-function is monotone

so

$$\frac{p_1(x)}{p_0(x)} > k'$$

is equivalent

$$x > k$$

to obtain k from restriction

$$P_0(\bar{X} > k) = \alpha \quad (= E_0 \varphi(\bar{x}))$$

Most powerful test is

$$\varphi(x) = \begin{cases} 1 & p_1(x) > k' p_0(x) \\ 0 & p_1(x) \leq k' p_0(x) \end{cases}$$

$$= \begin{cases} 1 & x > K \\ 0 & x \leq k \end{cases}$$

(18)

i.e.

$$\sigma K = z_\alpha ;$$

$$K = \sigma z_\alpha$$

