

MRE estimation (cont.)

Lecture 18

Ex 22 (cont.)

(i) $p(u) = u^2$, then
 $\hat{v} = E_0(\bar{x})$

so that $\hat{\delta} = x - E_0(\bar{x})$

is the MRE

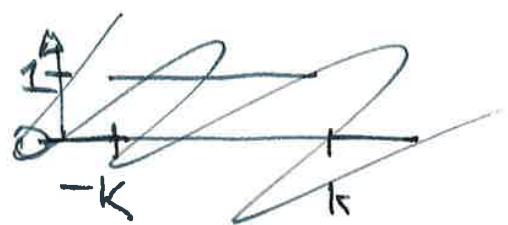
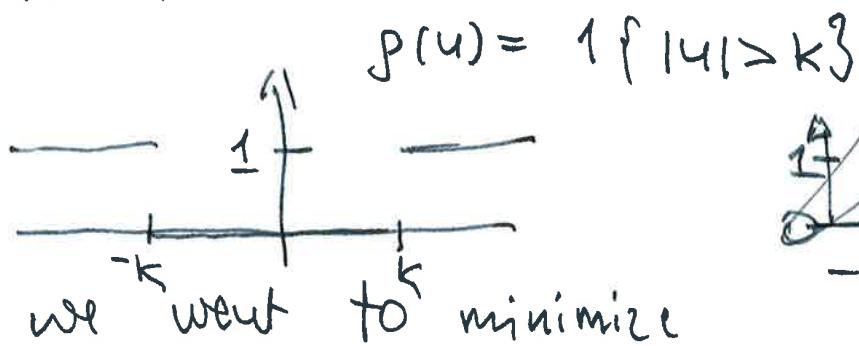
(ii) $p(u) = |u|$, then
 $\hat{v} = \text{med}(\bar{x})$

and

$$\hat{\delta} = x - \text{med}(\bar{x})$$

is the MRM.

(iii) If



$$\begin{aligned} E_0(g(\bar{x} - v)) &= E_0(1\{|\bar{x} - v| > k\}) \\ &= P_0(|\bar{x} - v| > k) \end{aligned}$$

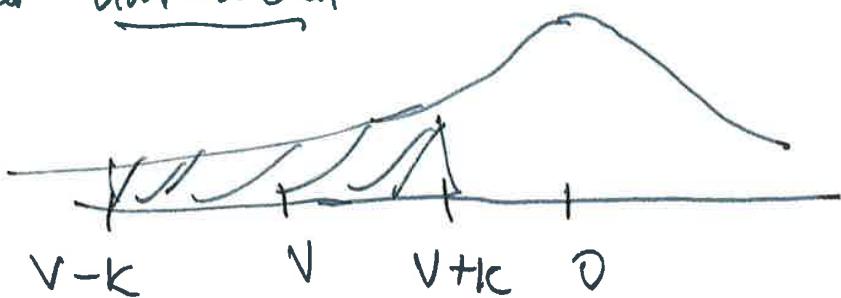
∴ for some v maximizing the

(1)

the complement, i.e. maximizing

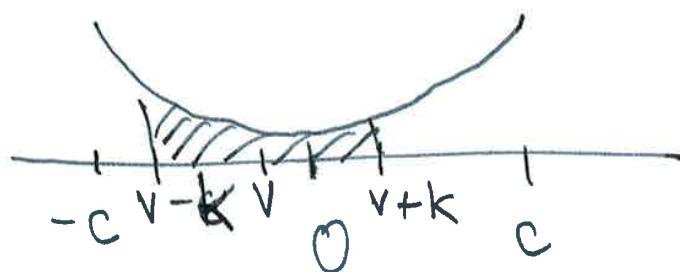
$$(1) P_0(|\bar{X} - v| \leq k) = P_0(v-k \leq \bar{X} \leq v+k)$$

- a) F_0 has density f_0 and symmetric around 0 and unimodal



then the probability (1) is maximized for $\hat{v} = 0$.

- b) F_0 has density f_0 symmetric around 0 and not U-shaped



$k < c$ is assumed, then either of

$$\begin{array}{ll} -c = v - k & v = \cancel{c} \\ c = v + k & v = c - k \end{array}$$

so that there are two ~~minima~~ ^{maxima} (MRE is not unique) (2)

$$\begin{aligned} \tilde{\delta}_1(x) &= x - k + c \\ \tilde{\delta}_2(x) &= x + k - c \end{aligned}$$

#1

Ex. x_1, \dots, x_n i.i.d. observations $N(\xi, \sigma^2)$
 σ^2 known. We see that

$$\tilde{\delta}_0(x) = \bar{x}$$

is equivariant, and we know it is a complete sufficient statistic. Let

$Y = (\bar{x}_1 - \bar{x}_n, \bar{x}_2 - \bar{x}_n, \dots, \bar{x}_{n-1} - \bar{x}_n)$ be a r.v., note that Y has a distribution that does not depend on ξ . Berly's theorem says that then \bar{x} (complete, sufficient statistic) is independent of Y (sufficiency). If we let

$p(u) = u^2$ so that $L(\xi, \theta) = (\theta - \xi)^2$
 Then

~~$$\hat{\delta}(\theta) = \arg \min_{\delta \in \mathbb{R}} E_P$$~~

$$\checkmark: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$$

$$\begin{aligned}
 \hat{J}(y) &= \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0(\rho(\delta_0(\bar{x}) - v(y)) \mid Y=y) \\
 &= \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0(\rho(\delta_0(\bar{x}) - v(y)) \mid Y=y) \\
 &= \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0(\rho(\delta_0(\bar{x}) - v(y))) \\
 &= \underset{\cancel{c: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}}{\operatorname{argmin}} \tilde{E}_0(\rho(\delta_0(\bar{x}) - c)) \quad (2) \\
 &\quad c \in \mathbb{R}
 \end{aligned}$$

Therefore

$$\hat{J}(x) = \bar{x} - \vec{v} \quad \text{constant in (2)}$$

is the MRE.

If in particular ρ is symmetric and convex, then since the distribution of \bar{x} is symmetric around 0 then we have $\vec{v} = 0$.

#

4

Theorem 7 (in lecture notes), read on your own

If we have squared error loss, the minimizer, $\hat{\delta}_0$, is

$$\hat{v}(y) = \bar{E}_0(\hat{\delta}_0(x) | Y=y)$$

so the MRE is

$$(3) \quad \hat{\delta}(x) = \hat{\delta}_0(x) - \bar{E}_0(\hat{\delta}_0(x) | Y=y)$$

Theorem (Pitman estimator)

Hypothesis X_1, \dots, X_n i.i.d. sample from a location family, with f density.

Let $Y = (X_1 - \bar{X}_n, \dots, X_{n-1} - \bar{X}_n)$, and

assume we have squared error loss

$$\mathbb{E} L(\hat{\delta}, d) = (\hat{\delta} - d)^2. \text{ Then the}$$

MRE is (3)

$$(5) \quad \hat{\delta}(x) = \frac{\int u f(x_1-u, \dots, x_n-u) du}{\int f(x_1-u, \dots, x_n-u) du}$$

Proof read on your own in lecture notes.
#1

Ex X_1, \dots, X_n i.i.d. $\text{Un}(\xi - b/2, \xi + b/2)$

b known, ξ unknown. The joint density
of X_1, \dots, X_n is

$$f(x_1 - \xi, \dots, x_n - \xi) = \begin{cases} \frac{1}{b^n} & \text{if } \xi - \frac{b}{2} \leq x_1 \leq \dots \leq x_n \leq \xi + \frac{b}{2} \\ 0 & \text{otherwise} \end{cases}$$

We use the Pitman estimator

$$\hat{\xi}(x) = \int_{x_{(n)}}^x u b^{-n} du$$

to be continued.

Randomized estimators and equivalence

Suppose $\tilde{J}(\bar{x})$ randomized estimator based on \bar{x} . Means that

$$\tilde{J}(\bar{x}) = \tilde{J}(\bar{x}, w)$$

r.v. diff.
 does not
 depend
 on the per. of
 interc.

deterministic

Defining invariance of location function as before

$$f(x, \xi) = f(x', \xi')$$

$$\lambda(\xi, d) = \lambda(\xi', d')$$

under transformations

$$x' = x + c$$

$$\xi' = \xi + c$$

$$d' = d + c$$

A randomized estimator is called equivariant if

$$\delta(\bar{x} + a, w) = \delta(\bar{x}, w) + a$$

As before one can show that bias, variance and risk of equivariant randomized estimators do not depend on the location parameter.

As before, the set of equivariant est.

is given

$$\left\{ \begin{array}{l} \delta(x, w) = \delta_0(x, w) + u(x, w) : \text{fixed est. est.} \\ u(x+a, w) = u(x, w) + a \end{array} \right\} \text{invariant est.}$$

and also, equivalently,

$$\delta(x, w) = \delta_0(x, w) - v(y, w)$$

$$y = (x_1 - x_0, x_2 - x_1, \dots, x_{n-1} - x_n)$$

To find the MRE, we minimize

$$E_0(\rho(\delta_0(\bar{X}, W) - v(Y, W)) | Y=y, W=w)$$

over the set of all v 's as above.

But we can start with an unnormalized estimator, $\delta_0(\bar{X})$. Then we want to minimize

$$E_0(\rho(\delta_0(\bar{X}) - v(Y, W)) | Y=y, W=w)$$

$$= E_0(\rho(\delta_0(\bar{X}) - v(y, w)) | Y=y, W=w)$$

$$\underset{\bar{X}, W \text{ independent}}{\rightarrow} = E_0(\rho(\delta_0(\bar{X}) - v(y, w)) | Y=y)$$

and the minimizer will not depend on w , only on y . Therefore, the MRE if it exists, will be unnormalized.

Ex. 25 read on your own

sufficiency and equivalence

read on your own.

Seale and location-seale model

read briefly on your own.