

Minimum risk equivariant estimators, cont.Lemma

Let δ_0 fixed equivariant estimator. The set of all ~~all~~ location equivariant estimators is given by

$$\Delta = \{ \delta = \delta_0 + u : u(x) = u(x+a), \text{ for all } x \in X, \text{ all } a \in \mathbb{R} \} \quad (1)$$

Proof.

(i) Let δ_0 be fixed equivariant, and u an invariant estimator $u(x+a) = u(x)$ and form the estimator

$$\delta = \delta_0 + u.$$

The

$$\begin{aligned} \underline{\delta(x+a)} &= \underline{\delta_0(x+a)} + \underline{u(x+a)} \\ &= \underline{\delta_0(x)} + a + \underline{u(x)} \\ &= \underline{\delta(x)} + a \end{aligned}$$

and thus $\delta = \delta_0 + u$ is equivariant

\Rightarrow therefore clear in (1)

(ii, \leq) Now let δ_0 be a fixed equivariant est. and $\delta \in \Delta$ (equivariant) arbitrary, and let $u = \delta - \delta_0$. Then

$$\begin{aligned} u(x+a) &= \delta(x+a) - \delta_0(x+a) \\ &= \delta(x) + a - (\delta_0(x) + a) \\ &= \delta(x) - \delta_0(x) \\ &= u(x) \end{aligned}$$

and thus $u(x)$ is invariant and $\delta = \delta_0 + u$

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Recall $X = \mathbb{R}^n$, $\Omega = \mathbb{R}$. The set of all invariant estimators

$$\mathcal{U} = \{u: u(x+a) = u(x), x \in X, a \in \mathbb{R}\}$$

Lemma.

If $X = \mathbb{R}^n$, $\Omega = \mathbb{R}$ then

$$\mathcal{U} = \{u: u(x_1, \dots, x_n) = h(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n), \text{ for } h \text{ function } \mathbb{R}^{n-1} \rightarrow \mathbb{R}\}$$

②

Proof

(\Leftarrow) Suppose that $u = h$ for a function
 $h: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ as above. Then,

$$\begin{aligned}
 u(x+a) &= h(x_1 + a - (x_n + a), x_2 + a - (x_n + a), \dots \\
 u(x_1 + a, \dots, x_n + a) &\quad \dots, x_{n-1} + a - (x_n + a)) \\
 &= h(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n) \\
 &= \underline{u(x_1, \dots, x_n)}
 \end{aligned}$$

and $u \in \mathcal{U}$ and $\Leftarrow \Rightarrow$ is proved.

(\Leftarrow) $u \in \mathcal{U}$ (invariant) so that

$$u(x+a) = u(x) \text{ all } x \in X = \mathbb{R}^n, a \in \mathbb{R}$$

Then

$$\begin{aligned}
 u(x_1, \dots, x_n) &= u(x) = u(\overset{\in \mathbb{R}^n}{x} - \overset{\in \mathbb{R}}{x_n}) = \\
 &= u(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n, x_n - x_n) \\
 &= \underbrace{u(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n)}_u, 0
 \end{aligned}$$

so that \Leftarrow is proved

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③

Note that if $n=1$ ($\gamma = \mathbb{R}$), then μ is invariant i.f.t. it is function of $x_1, x_1 = 0$, i.e. i.f.t. it is a constant

Note if we combine the previous two terms, we

Theorem Let δ_0 be a fixed equivariant estimator. Then

$$\Delta = \{ \bar{\delta} = \delta_0 - v : v \text{ function defined on } \mathbb{R}^{n-1} \} \#$$

Def. Suppose we have a location invariant estimation problem. If

$$\hat{\delta} = \underset{\delta \in \Delta}{\operatorname{argmin}} R(f, \delta)$$

exists, we call it the minimum risk equivariant estimator, (MRE), of f .

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⑨

Note: that since the risk of an equivariant estimator does not depend on f , we have that if

$$\hat{\delta} = \underset{\delta \in \Delta}{\operatorname{argmin}} R(\hat{f}, \delta)$$

exists, for some fixed \hat{f} , then $\hat{\delta}$ minimizes the risk over all Δ uniformly in \hat{f} .

Note also that since f is not important we can define the MRE, if it exists, as

$$\hat{\delta} = \underset{\delta \in \Delta}{\operatorname{argmin}} R(0, \delta)$$

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We can also characterize the set of all invariant loss functions ~~as~~ \mathbb{L} .

Lemma.

The set of all invariant loss functions \mathbb{L} , is given as

$$\mathbb{L} = \left\{ L(f, d) = \rho(d - f) : \rho \text{ a P function from } \mathbb{R} \text{ to } \mathbb{R}_+, \rho(0) = 0 \right\}$$

(5)

Proof.

(\Leftarrow) Suppose L invariant

$$L(\xi + a, d + a) = L(\xi, d)$$

Define $p(u) = L(0, u)$. Then

$$L(\xi, d) = L(0, d - \xi) = p(d - \xi)$$

(\Rightarrow) Suppose $L(\xi, d) = p(d - \xi)$. Then

$$\begin{aligned} L(\xi + a, d + a) &= p((d + a) - (\xi + a)) \\ &= p(d - \xi) = L(d, \xi) \end{aligned}$$

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Theorem

Suppose $\bar{x} = (x_1, \dots, x_n)$ distributed according to a location family, and define the vector $y = (y_1, \dots, y_{n-1})$ where

$$y_i = x_i - \bar{x}_n \quad , \quad i = 1, \dots, n-1 .$$

Suppose there is a (fixed) eigenvalue estimator $\hat{\xi}_n$ of ξ , with finite risk.

⑥

Then, if

$$\hat{v}(y) = \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0 (\rho(\delta_0(x) - v(y))) \mid Y=y$$

exists for all y , then

$$\hat{\delta}(x) = \delta_0(x) - \hat{v}(y)$$

is the MRE.

Proof.

The MRE is defined as

$$\hat{\delta} = \underset{\delta \in \Delta}{\operatorname{argmin}} R(0, \delta)$$

$$= \underset{\delta \in \Delta}{\operatorname{argmin}} E_0 (\lambda(0, \delta(x)))$$

by characterization
of invariant
~~of~~ function = ~~$\delta_0 - \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0 (\rho(\delta_0(x) - v(y)))$~~

$$\text{I.e.) function} = \underset{\delta \in \Delta}{\operatorname{argmin}} E_0 (\rho(\delta(x)))$$

by choice $\rightarrow = \delta_0 - \underset{v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} E_0 (\rho(\delta_0(x) - v(y)))$
of the set of equivariant estimators

(7)

$$= \delta_0 - \underset{v \in \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} \int E_0 (f(\delta_0(x) - v(y)) \mid Y=y) \\ \cdot dF_Y(y)$$

$$= \delta_0 - \underset{v \in \mathbb{R}^{n-1} \rightarrow \mathbb{R}}{\operatorname{argmin}} \int E_0 (\rho(\delta_0(x) - v(y)) \mid Y=y) dF_Y(y)$$

We minimize the integral by minimizing
the integral

$$\tau_0 (\rho(\delta_0(x) - v(y)) \mid Y=y)$$

for each y .

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Corollary.

Under the assumptions of the previous theorem, we have

(i) $\rho(u) = u^2$ (quadratic loss) then the MRE
 $\hat{\delta}(x) = \delta_0(x) - \bar{E}_0(\delta_0(\xi) | Y=y)$

(ii) $\rho(u) = |u|$ (absolute loss), then the
MRE $\hat{\delta}(x) = \delta_0(x) - \text{med}(\delta_0(\xi) | Y=y)$,

where $\text{med}(\delta_0(\xi) | Y=y)$ is the conditional median of $\delta_0(\xi)$ given that $Y=y$.

Proof

(i) We want to minimize

$$\bar{E}_0((\delta_0(\xi) - v(y))^2 | Y=y)$$

over v , and this is given by

$$\hat{v}(y) = \bar{E}_0(\delta_0(\xi) | Y=y)$$

⑨

(ii) We want to minimize

$$E_0(|\delta_0(x) - v(y)| \mid Y=y)$$

and this is obtained by

$$\hat{v}(y) = \text{med}(\delta_0(x) \mid Y=y).$$

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Ex. Suppose $n=1$. Then an arbitrary equivariant estimator $\bar{\delta}$ can be written as

$$\bar{\delta}(x) = \delta_0(x) + c$$

for a fixed equivariant estimator and an arbitrary constant $c \in \mathbb{R}$.

We see that $\delta_0(x) = x$ is equivariant, therefore an arbitrary estimator $\bar{\delta}$ can be written

$$\bar{\delta}(x) = x + c$$

Now we want to find the MRF!

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We need to find

$$\hat{v} = \underset{v \in R}{\operatorname{argmin}} E_0(p(\bar{x}-v))$$

If p is convex this is possible,
and if p strictly convex \hat{v} is unique
:

(11)