

Equivariance (MRE, minimum risk equivariant estimators)

Location models, as an illustrative example.

Suppose we have a location family of distributions

$$\{f(x; \xi), \xi \text{ location parameter}\}$$

$$f(x; \xi) = f(x_1 - \xi_1, \dots, x_k - \xi_k), \quad \xi = (\xi_1, \dots, \xi_k)$$

Loss function for measuring the loss in the estimation $L(\xi, d)$

ξ estimator value
 d parameter value

If we make a transformation

$$x \rightarrow x' = x + a$$

$$\xi \rightarrow \xi' = \xi + a$$

$$d \rightarrow d' = d + a$$

and then obtain

$$f(x'; \xi') = f(x, \xi) \quad (1)$$

$$L(\xi', d') = L(\xi, d) \quad (2)$$

Location invariance. By this setup
we have a location invariant estimation $\textcircled{1}$

problem.

The principle of equivariance

Suppose $\mathcal{F} = \{P_\theta : \theta \in \mathbb{R}\}$ a family of distributions $\mathcal{G} = \{g\}$ class of transformations of the sample space, suppose \mathcal{G} is a group under group operation equal to composition.

By

$$P_{\theta}(A).$$

we mean

$$\int_A dF_\theta(x) = P_{\theta}(A)$$

Def. \mathcal{F} invariant under \mathcal{G} , if

(i) $X \sim P_\theta$ and we make transformation

$$X' = gX \sim P_{\theta'} \in \mathcal{F}$$

for some $\theta' \in \mathbb{R}$

(ii) If \mathcal{G} fixed, let θ vary in \mathbb{R}
then so does $g\theta$. (traverser)

(2)

Ex. F d.f. fixed on \mathbb{R} , let
 $\bar{F} = \{F(x-\theta) : \theta \in \mathbb{R}\}$ (location family)
 and introduce group

$$G = \{g_\theta : \theta \in \mathbb{R}\}$$

with

$$g_\theta(x) = x + \theta.$$

Then \bar{F} is invariant under G . $\#$

Now suppose F is invariant under G .
 Then

$$P_\theta \sim \bar{x} \rightarrow g\bar{x} \sim P_{\theta'}$$

induces a map on Ω , let us call it
 \bar{g}

$$\bar{g}: \Omega \ni \theta \rightarrow \theta' \in \Omega$$

$$\text{or, } \theta' = \bar{g}\theta.$$

(Can argue that if in $\{P_\theta : \theta \in \Omega\}$
 different θ' 's give rise to different $P_{\theta'}$)

the the set

$$\{\bar{g} : g \in G\} = \bar{G}$$

is a group.

Ex. (ctd.) we have

$$g_a(x) = x + a$$

$L = \mathbb{R}$, the induced map is

$$F_G \cup \bar{x} \rightarrow \bar{x} + a \cup F_{G'}$$

then

$$\theta' = \theta + a$$

i.e.

$$\bar{g} : G \rightarrow \theta + a$$

or. $\theta' = \bar{g}\theta$.

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Relationship between G, \bar{G}

$$P_G(g \bar{x} \in A) = P_{G'}(\bar{x} \in A)$$

$$\text{or } P_G(\bar{x} \in f^{-1}(A))$$

④

$$P_\theta(g^{-1}(A)) = \underline{P}_{\bar{g}\theta}(A)$$

Suppose we have $\bar{f}, \bar{g}, \bar{h}$, we want to estimate $h(\theta)$ (estimated)

(Ex 21 in lecture notes, interesting, read on your own), to see that. . .

When we make a transformation, of θ , h should depend on θ only through $h(\theta)$, so

$$h(\bar{g}\theta)$$

should depend on θ only through $h(\theta)$.
Allow only two types of estimators.

Then, look at

$$\mathcal{H} = \{h(\theta) : \theta \in \mathbb{R}\}$$

vector space. The map \bar{g} induces a map $\bar{g}: \mathcal{H} \rightarrow \mathcal{H}$

$$\bar{g}: \theta \rightarrow \theta'$$

$$g^*: H \ni h(\theta) \rightarrow h(\bar{g}\theta) \in H$$

i.e.

$$g^* h(\theta) = h(\bar{g}\theta)$$

The condition that $h(\bar{g}\theta)$ depends on θ only through $h(\theta)$ makes g^* well defined, and $G^* = \{g^*: \bar{g} \in G\}$ is a group.

Now we have G, \bar{G}, G^* groups, & with the above invariance assumptions.

We call the loss function invariant if

$$L(\bar{g}\theta, g^*d) = L(\theta, d)$$

Now suppose G, \bar{G}, G^* groups as above & family of distributions, & invariant loss function. Then inference in ~~the~~ such a setting is called invariant.

⑥ "Invariant estimation problem."

Def An estimator $\delta(\bar{x})$ for an invariant estimation problem is called equivariant if

$$\delta(g\bar{x}) = g^*\delta(\bar{x})$$

for every $g \in G$, $g^* \in G^*$.

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Location equivalence

Location family

$$F = \{f(x-\xi) : \xi \in \mathbb{R}\}$$

for instance $\forall x \in \mathbb{R}^n$, $\mathbb{R} = \mathbb{R}$.

Def Location family, $L(\xi, d)$ a loss function, is called location invariant if

$$L(\xi + a, d) = L(\xi, d)$$

for all $\xi \in \mathbb{R}$, $d, a \in \mathbb{R}$. If so

the inference problem is called location invariant.

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Def. If δ location, L invariant loss function, we call δ estimator location equivariant

$$\delta(x+a) = \delta(x) + a$$

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Recall

$$f_{f+a}(x+a) = f_f(x) \quad \text{by construction}$$

of f

$$g_a(x) = x+a$$

~~$$g_a(z) = z+a$$~~

$$\bar{g}_a(\theta) = \theta+a$$

$$g_a^*(d) = d+a$$

Theorem Suppose we have location invariant problem, and δ is equivariant estimator. Then the bias, the variance of and the risk of δ do not depend

on the parameter γ (i.e. they are constant).

Proof.

(i) Bias

$$\begin{aligned} E_{\gamma}(\delta(X)) - \gamma &= E_0(\delta(X+\gamma)) - \gamma \\ &= E_0(\delta(X) + \gamma) - \gamma \\ &= E_0(\delta(X)) + \gamma - \gamma = E_0(\delta(X)) \end{aligned}$$

does not depend γ !

(ii) Variance

$$\begin{aligned} \text{Var}_{\gamma}(\delta(X)) &= E_{\gamma}(\delta^2(X)) - (E_{\gamma}(\delta(X)))^2 \\ &= E_0(\delta^2(X+\gamma)) - (E_0(\delta(X+\gamma)))^2 \\ &= E_0(\delta^2(X) + 2\delta(X)\gamma + \gamma^2) - \\ &\quad \cancel{(E_0(\delta(X)) + \gamma)} \\ &\quad \underbrace{(E_0(\delta(X)) + \gamma)^2}_{E_0(\delta(X)) + \gamma} \end{aligned}$$

(9)

$$= E_0(\delta^2(X)) + 2\int E_0(\delta(X)) + \xi^2 \\ - (E_0(\delta(X)))^2 - 2\int E_0(\delta(X)) - \xi^2$$

$$= \text{Var}_0(\delta(X))$$

does not depend on ξ .

(iii) The risk

$$\begin{aligned} E_\xi(L(\xi, \delta(X))) &= E_0(L(\xi, \delta(X+\xi))) \\ &= E_0(L(\xi, \delta(X)+\xi)) \\ &\xrightarrow{\text{invariance}} = E_0(L(0, \delta(X))) \end{aligned}$$

of the loss function
and thus not depend on ξ .

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(10)