

To prove that

Complement

$$E((\bar{X} - a)^2) = \varphi(a)$$

is minimized by $\hat{a} = E(\bar{X})$. By definition

$$\varphi(a) = \int (x-a)^2 dF(x)$$

where F is the distribution function of \bar{X} ,
meaning

$$\int (x-a)^2 dF(x) = \begin{cases} \int (x-a)^2 f(x) dx \\ \sum_k (k-a) f(k) \end{cases}$$

in the continuous and discrete r.v.'s cases,
respectively. Differentiation (allowed since everything)

$$\varphi'(a) = - \int 2(x-a) dF(x)$$

and solving

$$0 = \varphi'(a) = -2 \int (k-a) dF(x)$$

if $V\sigma(\bar{X})$
exists.

for a , we get

$$\text{if } 0 = \int x dF(x) - a \underbrace{\int dF(x)}_{=1}$$
$$= \int x dF(x) - a$$

i.e.

$$\hat{a} = \int x dF(x)$$

Note that this is a minimum since
 $\varphi(a)$ is convex in a .

To show that

$$E(|X-a|) = \psi(a)$$

is minimized by

$$\text{median}(X) = \tilde{x} = x_{\text{med}} = x_{0.5}$$

i.e. by the point ~~such that~~ $x_{0.5}$ such that

$$P(X \geq x_{0.5}) = 0.5$$

For simplicity in the following, to avoid taking generalised inverse in the case of a discrete r.v. (or any ~~not~~ point between two points strictly the median) let us assume that the r.v. X is continuous. Then

$$\begin{aligned}\psi(a) &= \int_{-\infty}^{\infty} |x-a| f(x) dx \\ &= \int_a^{\infty} (x-a) f(x) dx + \int_{-\infty}^a -(x-a) f(x) dx \\ &= \int_a^{\infty} x f(x) dx - \int_{-\infty}^a x f(x) dx \\ &\quad + a \left(\int_{-\infty}^a f(x) dx - \int_a^{\infty} f(x) dx \right)\end{aligned}$$

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Differentiating and solving $\psi'(a) = 0$:

$$\begin{aligned}0 &= \psi'(a) = -a f(a) - a f(a) \\&\quad + \left(\int_{-\infty}^a f(x) dx - \int_a^{\infty} f(x) dx \right) + a(f(a) + f(a)) \\&= \int_{-\infty}^a f(x) dx - \int_a^{\infty} f(x) dx = P(\bar{X} \leq a) \\&\quad - P(\bar{X} > a)\end{aligned}$$

i.e.

$$P(\bar{X} \leq a) = P(\bar{X} > a)$$

~~a.s.t~~ \Leftrightarrow

$$1 - P(\bar{X} > a) = P(\bar{X} > a)$$

$$\Leftrightarrow P(\bar{X} > a) = \frac{1}{2}$$

$$\Leftrightarrow a = x_{0.5} \text{ the median.}$$

To show that

$$E((\bar{X} - a)^2 | Y = b)$$

is minimized by

$$a = E(\bar{X} | Y = b)$$

(Deviation +) (abstract)
A quantity, the conditional expectation
 $E(\bar{X} | Y)$ is defined as L^2 -projection of
 \bar{X} on the σ -algebra generated by Y
 $\mathcal{F} = \sigma(Y)$, so that

$$E(\bar{X} | Y) = \underset{h(Y)}{\operatorname{argmin}} E((\bar{X} - h(Y))^2)$$

h measurable
function

Therefore

$$E((\bar{X} - a)^2 | Y) = E((\bar{X} - E(\bar{X} | Y))^2)$$

is minimized by (in terms of function
of Y)

~~$$E((\bar{X} - a)^2 | Y)$$~~ $a = E(\bar{X} | Y)$

Derivation 2, elementary.

$$(1) \quad E((X-a)^2 | Y=y) = \int (x-a)^2 dF_{X|Y=y}(x)$$

where $F_{X|Y=y}$ is the conditional distribution function of X given $Y=y$, i.e.

$$F_{X|Y=y}(x) = P(X \leq x | Y=y)$$

The differentiating (1) with respect to a , which is allowed if the integral exists for all a , which we assume (or at least in a neighbourhood around ~~around~~ $E(X|Y)$), we obtain and setting equaling to zero and solving,

$$0 = -2 \int (x-a) dF_{X|Y=y}(x)$$

\Leftrightarrow

$$0 = \int x dF_{X|Y=y}(x) - a \underbrace{\int dF_{X|Y=y}(x)}_{=1}$$

\Leftrightarrow

$$\begin{aligned} a &= \int x dF_{X|Y=y}(x) \\ &= E(X|Y=y). \end{aligned}$$

This is a minimum since (1) is convex in a .

To show that

$$E(|\bar{X} - a| | Y = y)$$

is minimized by

$$a = \text{med}(\bar{X} | Y = y)$$

= the conditional median of \bar{X}
given $Y = y$

one may simply copy the proof of showing
that

$$E(|\bar{X} - c|)$$

is minimized by the median $x_{0.5^-}$ for
the case where X and Y are continuous.
(The other case we except).

The bias rule is defined as

$$\int R(\theta, \delta) d\lambda(\theta) = r_\lambda(\delta)$$

where λ is the distribution function on \mathbb{R}^2 ,
the (prior) distribution of Θ . We can
also write this as

$$\begin{aligned} & \int E_\theta(L(\theta, \delta(\bar{x}))) d\lambda(\theta) \\ &= E(E(L(\theta, \delta(\bar{x}))) | \mathcal{W}) \end{aligned}$$

since if we see θ as a realization
of the r.v. \mathcal{W} then

$$\begin{aligned} R(\theta, \delta) &= E_\theta(L(\theta, \delta(\bar{x}))) = \cancel{E(L(\theta, \delta(\bar{x})) | \theta)} \\ &= E(L(\theta, \delta(\bar{x})) | \mathcal{W} = \theta) \end{aligned}$$

Thus

$$r_\lambda(\delta) = E(E(L(\theta, \delta(\bar{x}))) | \mathcal{W})$$

Then of course

$$\begin{aligned} E(E(g(\mathcal{W}, \bar{x})) | \mathcal{W}) &= E(g(\mathcal{W}, \bar{x})) \\ &= E(E(g(\mathcal{W}, \bar{x})) | \bar{x}) \end{aligned}$$

inner expectation
The last displayed formula is obtained
by interpreting

with respect to $F(\bar{w}|\bar{x})$, which of course
exists and can be obtained ~~as~~ by
using Bayes' formula.