Solutions

1.

$$P(A\cap B^c) = 1 - P(A^c \cup B) = 1 - 0.5,$$
$$P(A) = P(A\cap\Omega) = P(A\cap(B\cup B^c)) = P((A\cap B)\cup(A\cap B^c)) = 0.2 + 0.5 = 0.7,$$
$$P(A^c) = 1 - P(A) = 1 - 0.7 = 0.3.$$

2. Let $A =\{$ the deer does not come back$\}$, $H_1 =\{$the deer choses an apple$\}$, $H_2 =\{$the deer chooses a leaf$\}$. We are given the follwing

$$P(A|H_1) = 0.02, P(A|H_2) = 0.01, P(H_1) = 2/3, P(H_2) = 1/3.$$

The demanded probability is

$$P(H_1|A) = \frac{P(H_1A)}{P(A)} = \frac{P(A|H_1)P(H_1)}{P(A|H_1)P(H_1) + P(A|H_2)P(H_2)} = \frac{0.02\cdot 2/3}{0.02\cdot 2/3 + 0.01\cdot 1/3} = \ldots$$

3. Using the convolution formula we get

$$\begin{aligned} f_{U+V}(z) &= \int f_U(z-v)f_V(v)dv = \int \frac{1}{\theta_1}e^{-(z-v)/\theta_1}1\{z-v\geq 0\}\frac{1}{\theta_2}e^{-v/\theta_2}1\{v>0\}dv \\ &= e^{-z/\theta_1}\frac{1}{\theta_1\theta_2}\int_0^z e^{-v\frac{\theta_1-\theta_2}{\theta_1\theta_2}}dv = -e^{-z/\theta_1}\frac{1}{\theta_1-\theta_2}e^{-v\frac{\theta_1-\theta_2}{\theta_1\theta_2}}|_{v=0}^z = \ldots \end{aligned}$$

4. (a) Note first that the new r.v. U only takes values that are ≥ 0. The distribution function at a point $u\geq 0$ is

$$\begin{aligned} F_U(u) &= P(U\leq u) = P(-\log(1-X)\leq u) = P(\log(1-X)\geq -u) = P(1-X\geq e^{-u}) \\ &= P(X\leq 1-e^{-u}) = F_X(1-e^{-u}) = 1-e^{-u}. \end{aligned}$$

Thus U is a cont r.v. with probability density function $f_U(u) = e^{-u}$ for $u\geq 0$ and $f_U(u) = 0$ otherwise, thus U is $Exp(1))$ distributed.

(b) We have that

$$E(V) = E(Ye^{-Y^2}) = \int ye^{-y^2}f_Y(y)dy = \int_0^1 ye^{-y^2}dy = -\frac{1}{2}e^{-y^2}|_{y=0}^1 = \frac{1}{2}.$$

5. (a) We have that

$$0.5 = P(A) = (\text{cont of the probability measure}) = \lim_{n\to\infty}P(\cup_{i=1}^n A_j) = (\text{Boole's inequality})$$
$$\leq \lim_{n\to\infty}\sum_{i=1}^n P(A_j) = \lim_{n\to\infty}\sum_{i=1}^n p_j = \sum_{i=1}^\infty p_j.$$

Thus the restriction is that $\sum_{i=1}^\infty p_j \leq 0.5$.

(b) X_j are r.v.'s since the sets A_j are events, and they are independent since the A_j are independent. Thus the X_j are independent $Bern(p_j)$ distributed r.v.'s.

6. (a) The expectation and variance are

$$\begin{aligned}
E(F_n(t)) &= \frac{1}{n}\sum_{i=1}^{n} E(1\{X_i \le t\}) = \frac{1}{n}\sum_{i=1}^{n} P(X_i \le t) = \frac{1}{n}\sum_{i=1}^{n} F(t) = F(t). \\
\mathrm{Var}(F_n(t)) &= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(1\{X_i \le t\}) = \frac{1}{n^2}\sum_{i=1}^{n} E(1\{X_i \le t\}^2) - E(1\{X_i \le t\})^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} (E(1\{X_i \le t\}) - E(1\{X_i \le t\})^2) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} (F(t) - F(t)^2) \\
&= \frac{1}{n} F(t)(1 - F(t)).
\end{aligned}$$

We also have that since $nF_n(t)$ is a sum of independent Bernoulli($F(t)$) distributed r.v's, those are $1\{X_i \le t\}$ for $i = 1, \ldots, n$, therefore $nF_n(t)$ is distributed as $Bin(n, F(t))$-r.v.

(b) Since $F_n(t)$ is an average of independent identically distributed r.v.'s the CLT implies that the anwer is YES, with $\alpha(t) = F(t)$ and $\beta_n(t) = \sqrt{F(t)(1 - F(t))/n}$.