Mathematical Statistics **Exam 2017-11-13 at 8:00–13:00**
Centre for Mathematical Sciences **MASA01 A**
Lund University **Probability theory**

Complete solutions to all problems are required. Models should be specified, approximations and conclusions should be specified and motivated.

Problems 1, 2 and 3 can give a maximum of 10 points each. Problems 4, 5 and 6 can give a maximum of 20 points each. The maximum total on the exam is 90 points. The limit for Pass (P) is at least 45 points, and the limit for High Pass (HP) is at least 67 points.

Only the paper sheets provided by the institute in the exam room shall be used, both for solutions and for sketches/notes.

Each problem solution shall start at the top of a new paper.

Write your ID number on each paper that you hand in.

Red pencil or pen is not allowed.

Only statistical tables provided by the institute are allowed. Electronic calculators are allowed. No other tables or formula sheets are allowed.

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1. Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, and suppose that E, F are events. Suppose further that $P(E) = 0.2, P(F) = 0.3$, and that $P(E \cap F) = 0.1$ Calculate $P((E \cup F)^c)$. [10p]

2. Suppose that a migrating bird of a certain species that rests at the resting area Flommen in Falsterbo in Skåne, can be in three different states: Eating (E), looking for Food (F) and Socialising (S), meaning for instance protection of it's territory against other birds and avoiding predators. Any bird spends their time in the different states with a relative frequency of 20%(E), 40%(F) and 40%(S), respectively. The birds can be caught in nets, that were put on the resting area by ornithologists that want to study the behaviour of the birds. Suppose that the probabilities of getting caught in a net are 0.1 for the birds that eat, 0.5 for the birds that look for food and 0.3 for the birds that are in the socialising state, respectively.

   (a) What is the relative frequency of the birds that are caught in nets? [5p]

   (b) What is the proportion of birds that were looking for food of those that were caught in a net? [5p]

3. Let X_1 and X_2 be two independent r.v.'s with $X_1 \in Un([1, 5])$ and $X_2 \in Un([3, 7])$.

   (a) Derive the probability density function of $X_1 + X_2$. [5p]

   (b) Calculate $P(X_1 + 2X_2 \leq 13)$. [5p]

4. Suppose that an electrical current flows through a capacitor (it does not really flow through but one can imagine it when working out the formulas), that can store an electrical charge up to a certain maximum charge, which for a certain type of capacitors A is 1 mC (mini Coulomb). The current that flows through the capacitor can be thought of building up a total electrical charge Q (in mC) which can be seen as a r.v. with probability density function

$$f_Q(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

   if the capacitor would be an ideal capacitor, with no upper storage limit. Let $\tilde{Q}$ be the total charge that is stored in the capacitor of type A.

(a) What is the proportion of capacitors (of type A) that will be fully charged in the above setting? [4p]

(b) Find the density function $f_{\tilde{Q}}$ of $\tilde{Q}$, and state which measure μ that $f_{\tilde{Q}}$ is a density with respect to. [10p]

(c) Calculate $E(\tilde{Q})$. [6p]

5. A physics lab conducts experiments including radioactive particles. Suppose that the number of radioactive particles that decay in a material during a certain time $[0, t]$ can accurately be modeled as a Poisson distributed r.v. with expectation θ. It is known that the relative frequency of there being exactly zero particles that decay during $[0, t]$ is 0.1.

   (a) The physicists will call the experiment a success if there are two or more particles that decay during the time $[0, t]$. Calculate the probability of success. [10p]

   (b) Suppose that a large number of labs, say n, perform the same experiment, independently of each other, with the same parameters as in the lab above. Give an expression for the probability that at least one of them will report their experiment a success. What will this probability be if $n = 100$? [10p]

6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed r.v.'s, with expectation $E(X_i) = 0$ and $Var(X_i) = 1$. Define $U_i = X_i + X_{i+1}$, for $i = 1, 2, \ldots$.

   (a) Calculate $E(U_i), Var(U_i)$, and $Cov(U_i, U_{i+1})$. Show that $Cov(U_i, U_{i+k}) = 0$ for $k = 2, 3, \ldots$ and for any i. [10p]

   (b) Show that

$$\frac{U_1 + \ldots + U_n}{n} \xrightarrow{L^2} c$$

   as $n \to \infty$ for some c, and derive the value of c. [10p]