Mathematical Statistics **Exam 2016-11-21 at 8:00–13:00**
Centre for Mathematical Sciences **MASA01 A**
Lund University **Probability theory**

Complete solutions to all problems are required. Models should be specified, approximations and conclusions should be specified and motivated.

Problems 1, 2 and 3 can give a maximum of 10 points each. Problems 4, 5 and 6 can give a maximum of 20 points each. The maximum total on the exam is 90 points. The limit for Pass (P) is at least 45 points, and the limit for High Pass (HP) is at least 67 points.

Only the paper sheets provided by the institute in the exam room shall be used, both for solutions and for sketches/notes.

Each problem solution shall start at the top of a new paper.

Write your ID number on each paper that you hand in.

Red pencil or pen is not allowed.

Only statistical tables provided by the institute are allowed. No other tables or formula sheets are allowed.

The results of the exam will be posted at the latest 2016-11-30.

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1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that $A, B \in \mathcal{F}$ are such that

$$P(A|B) = 0.1, P(B^c) = 0.4, P(A \cup B) = 0.64.$$

Are A and B independent?

[10p]

2. Doctors at a childrens hospital, that are specialists in treating children born with heart problems, classify children that arrive to the hospital into one of three possible categories: Normal heart function (NHF), Arythmic heart function (AHF), and Repeated heart failure (RHF). The children are then given appropriate treatment, depending on which category they are classified into. The proportion of children taken to the hospital for treatment, that are classified into the NHF group is 60% and that are classified into the AHF group is 30 %. The proportions of children that live past their second birthday is, in the group NHF 90%, in the group AHF 20% and in the group RHF 5%.

   (a) What is the proportion of children taken to the hospital that live past their second year? [5p]

   (b) What is the proportion of children that were classified with RHF, in the group that did not live past their second birthday? [5p]

3. Suppose that X and Y are two independent r.v.'s with probability density functions

$$f_X(x) = \begin{cases} \frac{1}{5}, & 0 \le x \le 5, \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}y, & 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Define the r.v. $Z = X + Y$.

Turn page for rest of Problem 3, and Problems 4,5,6.

(a) Find the probability density function for Z. [5p]

(b) Find $E(Z^2)$. [5p]

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The r.v. U has density

$$f(u) = \begin{cases} \frac{5}{12}, & x = 0, \\ c, & x = 2, \\ \frac{1}{x^2}, & 3 \leq x \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

with respect to the measure $\mu(u) = \mu_0(u)1\{u < 3\} + \mu_1(u)1\{u \geq 3\}$, where μ_0 is the measure counting the integers in $\mathbf{R}$ and μ_1 is ordinary length measure on $\mathbf{R}$. The constant c is unknown.

(a) Determine c, and determine the d.f. F of U. [10p]

(b) Calculate $E(1/(X-1))$. [10p]

5. The game G is played as follows: A player P rolls a perfectly symmetric die, repeatedly, until the resulting number of dots shown on the die is 6. The player is successful (SF) if the number of rolls of the die needed, until one gets a 6, is exactly 2. Now suppose that the game is played by n players, $P_1, \ldots, P_n$, and the results of their dice rollings are independent. How many players have to play the game, in order for the probability that there will be least one successful player, will be greater than 0.8? [20p]

6. Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $X_1, \ldots, X_n$ are i.i.d. continuous random variables with d.f. F. Suppose that $g : \mathbf{R} \to \mathbf{R}$ is a continuous monotonically decreasing function, with continuous inverse g^{-1}, and define the r.v.'s $U_i = g^{-1}(X_i)$, for $i = 1, \ldots, n$. We assume that g^{-1} is such that both $E(U_i)$ and $E(U_i^2)$ exists.

(a) Are the r.v.'s $U_1, \ldots, U_n$ independent? Are they identically distributed? [6p]

(b) Derive an expression for the d.f. F_U of U. [4p]

(c) The empirical d.f. $F_U^{(n)}$, for the U_i's, is given as

$$F_U^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} 1\{U_i \leq t\}.$$

Suppose that t is a fixed number. Show that

$$F_U^{(n)}(t) \stackrel{L^2}{\to} H(t),$$

as $n \to \infty$, for some function H. Derive that function H. [10p]