1. Let \((\Omega, \mathcal{F}, P)\) be a probability space. Suppose that \(A, B \in \mathcal{F}\) are such that

\[
P(A|B) = 0.1, \quad P(B^c) = 0.4, \quad P(A \cup B) = 0.64.
\]

Are \(A\) and \(B\) independent? [10p]

2. Doctors at a children's hospital, that are specialists in treating children born with heart problems, classify children that arrive to the hospital into one of three possible categories: Normal heart function (NHF), Atrymic heart function (AHF), and Repeated heart failure (RHF). The children are then given appropriate treatment, depending on which category they are classified into. The proportion of children taken to the hospital for treatment, that are classified into the NHF group is 60% and that are classified into the AHF group is 30%. The proportions of children that live past their second birthday is, in the group NHF 90%, in the group AHF 20% and in the group RHF 5%.

   (a) What is the proportion of children taken to the hospital that live past their second year? [5p]
   (b) What is the proportion of children that were classified with RHF, in the group that did not live past their second birthday? [5p]

3. Suppose that \(X\) and \(Y\) are two independent r.v.’s with probability density functions

\[
f_X(x) = \begin{cases} \frac{1}{5}, & 0 \leq x \leq 5, \\ 0, & \text{otherwise} \end{cases}
\]

\[
f_Y(y) = \begin{cases} \frac{1}{2}y, & 0 \leq y \leq 2, \\ 0, & \text{otherwise} \end{cases}
\]

Define the r.v. \(Z = X + Y\).

Turn page for rest of Problem 3, and Problems 4,5,6.
(a) Find the probability density function for $Z$. [5p]
(b) Find $E(Z^2)$. [5p]

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The r.v. $U$ has density

$$f(u) = \begin{cases} \frac{5}{12}, & x = 0, \\ c, & x = 2, \\ \frac{1}{2}, & 3 \leq x \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

with respect to the measure $\mu(u) = \mu_0(u)1\{u < 3\} + \mu_1(u)1\{u \geq 3\}$, where $\mu_0$ is the measure counting the integers in $\mathbb{R}$ and $\mu_1$ is ordinary length measure on $\mathbb{R}$. The constant $c$ is unknown.

(a) Determine $c$, and determine the d.f. $F$ of $U$. [10p]
(b) Calculate $E(1/(X-1))$. [10p]

5. The game $G$ is played as follows: A player $P$ rolls a perfectly symmetric die, repeatedly, until the resulting number of dots shown on the die is 6. The player is successful (SF) if the number of rolls of the die needed, until one gets a 6, is exactly 2. Now suppose that the game is played by $n$ players, $P_1, \ldots, P_n$, and the results of their dice rollings are independent. How many players have to play the game, in order for the probability that there will be at least one successful player, will be greater than 0.8? [20p]

6. Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $X_1, \ldots, X_n$ are i.i.d. continuous random variables with d.f. $F$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous monotonically decreasing function, with continuous inverse $g^{-1}$, and define the r.v.’s $U_i = g^{-1}(X_i)$, for $i = 1, \ldots, n$. We assume that $g^{-1}$ is such that both $E(U_i)$ and $E(U_i^2)$ exists.

(a) Are the r.v.’s $U_1, \ldots, U_n$ independent? Are they identically distributed? [6p]
(b) Derive an expression for the d.f. $F_U$ of $U$. [4p]
(c) The empirical d.f. $F_U^{(n)}$, for the $U_i$’s, is given as

$$F_U^{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} 1\{U_i \leq t\}.$$  

Suppose that $t$ is a fixed number. Show that

$$F_U^{(n)}(t) \overset{L^2}{\to} H(t),$$

as $n \to \infty$, for some function $H$. Derive that function $H$. [10p]