1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are A, C, F. A spectral density is always positive, which is true for B, D, H. Thus E, G, I are realizations.

The covariance functions in A and C belong to AR-processes, as they certainly have covariance values for $\tau > 4$. A varies faster than C, which correspond to the strong spectral peak at $f = 0.5$ in D and the realization in E, where C corresponds to the spectral peak in B and the process realization in I. The orders are AR(1) for A, D, E and AR(2) (peaks in $f = 0.25$ and $f = -0.25$) for C, B, I. The covariance function in F is an MA(1)-process as it is zero for $\tau \geq 2$, and the smoother spectral density function in H and the irregular noisy realization in G.

2) We investigate the new process $Y(t) = X'(t) - X(t)$ and find $E[Y(t)] = 0 - 2 = -2$.

and

$V[Y(t)] = C[X'(t) - X(t), X'(t) - X(t)] = V[X'(t)] + V[X(t)] = -r_X'(0) + r_X(0)$.

With $r_X'(\tau) = -2\tau e^{-\tau^2}$, and $r_X''(\tau) = -2e^{-\tau^2} + 4\tau^2 e^{-\tau^2}$, the variance becomes $V[Y(t)] = 2 + 1 = 3$.

The probability

$P(Y(t) \geq 0) = 1 - \Phi\left(\frac{0 - (-2)}{\sqrt{3}}\right) = 1 - \Phi(1.15) = 1 - 0.875 = 0.125$.

3) a) Wrong, the Welch method has shorter windows and thereby the resolution is worse than the periodogram.

b) Correct, as we can see that the spectrum estimate level at $f > 0.3$ for B is much higher than in A.

c) Wrong, the overlap is 50% so the window lengths are $L = 32$.

d) Correct, the variance is reduced a factor $K$ where $K$ is the number of windows. The standard deviation is then reduced a factor $\sqrt{K} = \sqrt{4} = 2$.

e) Correct, as the resolution of peaks are worse in A and the leakage and bias at high frequencies is worse in B.

4) a) The AR-process has an exponentially decreasing covariance function and the MA-process has a covariance function that is zero for $|\tau| > 1$. The figure is then most likely to show the covariance estimate of the MA-process as the larger values of $\tau$ seems to vary around zero.

b) With the true covariance model of the MA-process, the variances of the two estimates $m_1^*$ and $m_2^*$ are

$V[m_1^*] = 10/9V[e(t)]$  $V[m_2^*] = V[e(t)]$.

Conclusion: The estimate $m_2^*$ has the smallest variance.

5 a) $R_Z(\nu) = \begin{cases} 2(1 - 8|\nu + \frac{3}{8}|) & \text{for } -0.5 < \nu \leq -0.25 \\ 0 & \text{for } -0.25 < \nu < 0.25 \\ 2(1 - 8|\nu - \frac{3}{8}|) & \text{for } 0.25 \leq \nu < 0.5 \end{cases}$
6) a) The Wiener filter is given from

\[ R_G(\nu) = 2(1 - 8|\nu|) \quad |\nu| \leq 0.125, \]

\[ r_G(n) = \begin{cases} \frac{1}{2} \left( \frac{32}{(2\pi h)^2} \right) (1 - \cos(\frac{2\pi n}{h})) & n = 0 \\ \frac{1}{2} & n = \pm 1, \pm 2, \pm 3, \ldots \end{cases} \]

giving \( r_Z(n) = 2r_G(n)\cos(2\pi \frac{3}{8}n) \) where \( n = 0, \pm 1, \pm 2, \ldots \).

b) The covariance function is given from the table of formulas as

\[ R_Z(\nu) = R_G(\nu - \frac{3}{8}) + R_G(\nu + \frac{3}{8}) \]

with

\[ R_G(\nu) = 2(1 - 8|\nu|) \quad |\nu| \leq 0.125, \]

\[ r_G(n) = \begin{cases} \frac{1}{2} \left( \frac{32}{(2\pi h)^2} \right) (1 - \cos(\frac{2\pi n}{h})) & n = 0 \\ \frac{1}{2} & n = \pm 1, \pm 2, \pm 3, \ldots \end{cases} \]

giving \( r_Z(n) = 2r_G(n)\cos(2\pi \frac{3}{8}n) \) where \( n = 0, \pm 1, \pm 2, \ldots \).

c) The spectral density is

\[ R_Y(f) = |H(f)|^2 R_X(f), \]

where \(|H(f)| = 1\) for \(|f| \leq 1\), yielding

\[ R_Y(f) = \begin{cases} 0 & \text{for } |f| < 0.75 \\ 4 - 4|f| & \text{for } 0.75 \leq |f| < 1 \\ 0 & \text{for } 1 \leq |f|. \end{cases} \]

6) a) The Wiener filter is given from

\[ H(f) = \frac{R_s(f)}{R_S(f) + R_N(f)}, \]

where

\[ R_S(f) = \sum_{\tau} r_S(\tau)e^{-i2\pi f\tau} = 2 - 2\cos(2\pi f), \]

and

\[ R_N(f) = \sum_{\tau} r_N(\tau)e^{-i2\pi f\tau} = 2 + 2\cos(2\pi f). \]

We get

\[ H(f) = 0.5 - 0.5\cos(2\pi f) = 0.5 - 0.25(e^{-i2\pi f} + e^{i2\pi f}) \quad -0.5 \leq f < 0.5 \]

We identify the impulse response as \( h(0) = 0.5, h(\pm 1) = -0.25 \) and zero for all other values of \( t \).

b) For the given format of the impulse response,

\[ \min_{a,b} E[(aX(t) + bX(t + 1) - S(t))^2], \]

\[ \min_{a,b} E[(aS(t) + 0.5aS(t - 1) + aN(t) + bS(t + 1) + 0.5bS(t) + bN(t + 1) - S(t))^2], \]

\[ \min_{a,b} ((a - 1 + 0.5b)^2 + 0.25a^2 + b^2)r_S(0) + (a^2 + b^2)r_N(0) + \]
\[ + 2(0.5a(a - 1 + 0.5b) + b(a - 1 + 0.5b))r_S(1) + 2abr_N(1)), \]

simplified into

\[ \min f(a, b) = \frac{7}{2}a^2 + \frac{7}{2}b^2 + \frac{3}{2}ab - 3a + 2. \]

Differentiation with respect to \( a \) and \( b \) gives

\[ \frac{\partial f}{\partial a} = 7a + \frac{3b}{2} - 3 = 0, \]
\[ \frac{\partial f}{\partial b} = 7b + \frac{3a}{2} = 0, \]

with \( b = -18/187 \approx -0.0963 \) and \( a = 84/187 \approx 0.449 \). The impulse response is \( h(0) = 0.449, h(-1) = -0.0963 \) and zero for all other values of \( t \).