Mathematical Statistics, Centre for Mathematical Sciences
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Solutions to exam in Stationary stochastic processes (FMSF10/MASC04)
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1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are A, C, F. A spectral density is always positive, which is true for B, D, H. Thus E, G, I are realizations.

The covariance functions in A and C belong to AR-processes, as they certainly have covariance values for $\tau > 4$. A varies faster than C, which correspond to the strong spectral peak at $f = 0.5$ in D and the realization in E, where C corresponds to the spectral peak in B and the process realization in I. The orders are AR(1) for A,D E and AR(2) (peaks in $f = 0.25$ and $f = -0.25$) for C, B, I. The covariance function in F is an MA(1)-process as it is zero for $\tau \geq 2$, and the smoother spectral density function in H and the irregular noisy realization in G.

2) We investigate the new process $Y(t) = X'(t) - X(t)$ and $P(Y(t) \geq 0)$, and find

$$E[Y(t)] = 0 - 2 = -2.$$

and

$$V[Y(t)] = C[X'(t) - X(t), X'(t) - X(t)] = V[X'(t)] + V[X(t)] = -r''_X(0) + r_X(0).$$

With $r'_X(\tau) = -2\tau e^{-\tau^2}$, and $r''_X(\tau) = -2e^{-\tau^2} + 4\tau^2 e^{-\tau^2}$, the variance becomes $V[Y(t)] = 2+1 = 3$. The probability

$$P(Y(t) \geq 0) = 1 - \Phi(\frac{0-(-2)}{\sqrt{3}}) = 1 - \Phi(1.15) = 1 - 0.875 = 0.125.$$

3) a) Wrong, the Welch method has shorter windows and thereby the resolution is worse than the periodogram.

b) Correct, as we can see that the spectrum estimate level at $f > 0.3$ for B is much higher than in A.

c) Wrong, the overlap is 50% so the window lengths are $L = 32$.

d) Correct, the variance is reduced a factor K where K is the number of windows. The standard deviation is then reduced a factor $\sqrt{K} = \sqrt{4} = 2$.

e) Correct, as the resolution of peaks are worse in A and the leakage and bias at high frequencies is worse in B.

4) a) The AR-process has an exponentially decreasing covariance function and the MA-process has a covariance function that is zero for $|\tau| > 1$. The figure is then most likely to show the covariance estimate of the MA-process as the larger values of τ seems to vary around zero.

b) With the true covariance model of the MA-process, the variances of the two estimates m_1^* and m_2^* are

$$V[m_1^*] = 10/9V[e(t)] \quad V[m_2^*] = V[e(t)].$$

Conclusion: The estimate m_2^* has the smallest variance.

5 a)

$$R_Z(\nu) = \begin{cases} 2(1 - 8(|\nu + \frac{3}{8}|) & \text{for } -0.5 < \nu \leq -0.25 \\ 0 & \text{for } -0.25 < \nu < 0.25 \\ 2(1 - 8(|\nu - \frac{3}{8}|) & \text{for } 0.25 \leq \nu < 0.5 \end{cases}$$

b) The covariance function is given from the table of formulas as $R_Z(\nu) = R_G(\nu - \frac{3}{8}) + R_G(\nu + \frac{3}{8})$ with

$$R_G(\nu) = 2(1 - 8|\nu|) \quad |\nu| \le 0.125,$$

$$r_G(n) = \begin{cases} \frac{1}{4} & n = 0 \\ \frac{32}{(2\pi n)^2}(1 - \cos(\frac{2\pi n}{8})) & n = \pm 1, \pm 2, \pm 3, \ldots \end{cases}$$

giving $r_Z(n) = 2 r_G(n) \cos(2\pi \frac{3}{8} n)$ where $n = 0, \pm 1, \pm 2 \ldots$.

c) The spectral density is

$$R_Y(f) = |H(f)|^2 R_X(f),$$

where $|H(f)| = 1$ for $|f| \le 1$, yielding

$$R_Y(f) = \begin{cases} 0 & \text{for } |f| < 0.75 \\ 4 - 4|f| & \text{for } 0.75 \le |f| < 1 \\ 0 & \text{for } 1 \le |f|. \end{cases}$$

6) a) The Wiener filter is given from

$$H(f) = \frac{R_s(f)}{R_S(f) + R_N(f)},$$

where

$$R_S(f) = \sum_\tau r_S(\tau) e^{-i2\pi f \tau} = 2 - 2\cos(2\pi f),$$

and

$$R_N(f) = \sum_\tau r_N(\tau) e^{-i2\pi f \tau} = 2 + 2\cos(2\pi f).$$

We get

$$H(f) = 0.5 - 0.5\cos(2\pi f) = 0.5 - 0.25(e^{-i2\pi f} + e^{i2\pi f}) \quad -0.5 \le f < 0.5$$

We identify the impulse response as $h(0) = 0.5$, $h(\pm 1) = -0.25$ and zero for all other values of t.

b) For the given format of the impulse response,

$$\min_{a,b} E[(aX(t) + bX(t+1) - S(t))^2],$$

$$\min_{a,b} E[(aS(t) + 0.5aS(t-1) + aN(t) + bS(t+1) + 0.5bS(t) + bN(t+1) - S(t))^2],$$

$$\min_{a,b}((a - 1 + 0.5b)^2 + 0.25a^2 + b^2) r_S(0) + (a^2 + b^2) r_N(0) +$$
$$+ 2(0.5a(a - 1 + 0.5b) + b(a - 1 + 0.5b)) r_S(1) + 2ab r_N(1)),$$

simplified into

$$\min f(a,b) = \frac{7}{2}a^2 + \frac{7}{2}b^2 + \frac{3}{2}ab - 3a + 2.$$

Differentiation with respect to a and b gives

$$\begin{aligned} \frac{\partial f}{\partial a} &= 7a + \frac{3b}{2} - 3 = 0, \\ \frac{\partial f}{\partial b} &= 7b + \frac{3}{2}a = 0, \end{aligned}$$

with $b = -18/187 \approx -0.0963$ and $a = 84/187 \approx 0.449$. The impulse response is $h(0) = 0.449$, $h(-1) = -0.0963$ and zero for all other values of t.