Correct solutions of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits. The maximal number of credits are 90. The limit for passing the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) a) Correct answers: B) C) E) H)  
   b) A-D-I AR(1), B-F-G AR(2), C-E-H MA(2)

2) 3,4,5,7 are correct, 1,2,6,8,9,10 are false

3) a) Wrong. $f_s \geq 6$ i.e. $d \leq 1/6$.  
   b) Wrong. The resulting spectral density is  
      
      $$R_Y(f) = 1, \quad 1 \leq |f| \leq 2.$$  
   
   c) Correct.

4) a) The Wiener filter is given from  
      
      $$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)}.$$  
      
      We find  
      
      $$R_S(f) + R_N(f) = \begin{cases} 1, & |f| \leq 1, \\ |f|, & 1 < |f| \leq 2, \end{cases}$$  
      
      and  
      
      $$\frac{R_S(f)}{R_S(f) + R_N(f)} = \begin{cases} |f|, & |f| \leq 1, \\ 1, & 1 < |f| \leq 2, \end{cases}$$  
      
      or according to the figure below.

The resulting SNR is  

$$\text{SNR} = \frac{\int R_S(f) df}{\int \frac{R_S(f)R_N(f)}{R_S(f) + R_N(f)} df} = \frac{2 \int_0^2 f df}{2 \int_0^1 f(1-f) df} = \frac{4}{1/3} = 12.$$  

The resulting $\text{SNR}_{opt} = 12$.

b) The SNR is defined as  

$$\text{SNR} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$  

where  

$$E[S^2(t)] = V[S(t)] = 2 \int_0^2 R_S(f) df = 4,$$
and
\[
E[(Y(t) - S(t))^2] = E[(S_{out}(t) - S(t))^2] + E[N_{out}^2(t)],
\]
\[
V[(S_{out}(t) - S(t))] + V[N_{out}(t)].
\]
where \(Y(t) = S_{out}(t) + N_{out}(t)\), and the cross-covariance term is discarded as \(S(t)\) and \(N(t)\) are independent. We find
\[
V[(S_{out}(t) - S(t))] = 2 \int_0^2 |H(f) - 1|^2 R_S(f) df = 2 \int_0^1 R_S(f) df = 1,
\]
and
\[
V[N_{out}(t)] = 2 \int_0^1 |H(f)|^2 R_N(f) df = 0.
\]
Finally, the resulting
\[
\text{SNR} = \frac{V[S(t)]}{V[(S_{out}(t) - S(t))] + V[N_{out}(t)]} = \frac{4}{1 + 0} = 4.
\]
5) a) Define
\[
\epsilon_{t+1} = X_{t+1} - Y_{t+1} = e_{t+1} + e_t + e_{t-1} - (e_t + e_{t-1} + e_{t-2}) = e_{t+1} - e_{t-2},
\]
where \(e_{t+1} \in N(0, V[e_{t+1} - e_{t-2}]) = N(0, 1/2)\) as \(e_{t+1}\) and \(e_{t-2}\) are independent, and as linear combinations of Gaussian processes also are Gaussian processes. Then
\[
P(\{|\epsilon_{t+1}| > 1\}) = 2P(e_{t+1} > 1) = 2(1 - P(e_{t+1} \leq 1)) = 2(1 - \Phi(\sqrt{2})) = 0.1586.
\]
b) For \(a\) and \(b\) generally we get
\[
\epsilon_{t+1} = e_{t+1} + e_t + e_{t-1} - a(e_t + e_{t-1} + e_{t-2}) - b(e_{t-1} + e_{t-2} + e_{t-3}),
\]
Minimizing \(P(\{|\epsilon_{t+1}| > 1\})\) is the same as minimizing,
\[
V[\epsilon_{t+1}] = V[e_{t+1} + (1 - a)e_t + (1 - a - b)e_{t-1} - (a + b)e_{t-2} - be_{t-3}],
\]
\[
= \frac{1}{4}(1 + (1 - a)^2 + (1 - a - b)^2 + (a + b)^2 + b^2),
\]
\[
= \frac{1}{4}(1 + (1 - 2a + a^2) + (1 - 2a - 2b + 2ab + a^2 + b^2) + (a^2 + 2ab + b^2 + b^2)).
\]
where \(e_t \in N(0, 1/4)\) are independent stochastic variables, or minimizing
\[
V[\epsilon_{t+1}] = (1 + a^2 + b^2)r_X(0) + 2a(b - 1)r_X(1) - 2br_X(2),
\]
where \(r_X(0) = \frac{3}{4}, r_X(1) = \frac{1}{2}\) and \(r_X(2) = \frac{1}{4}\). Simplifications give
\[
V[\epsilon_{t+1}] = \frac{1}{4}(3 + 3a^2 + 3b^2 + 4ab - 4a - 2b),
\]
and the partial derivatives of \(V[\epsilon_{t+1}]\) with respect to \(a\) and \(b\) are
\[
\frac{dV}{da} = \frac{1}{4}(6a + 4b - 4) = \frac{3}{2}a + b - 1 = 0,
\]
and
\[
\frac{dV}{db} = \frac{1}{4}(4a + 6b - 2) = a + \frac{3}{2}b - \frac{1}{2} = 0.
\]
The solution is found putting the first equation
\[
b = 1 - \frac{3}{2}a,
\]
into the second equation
\[
a = \frac{1}{2} - \frac{3}{2}(1 - \frac{3}{2}a) = \frac{1}{2} - \frac{3}{2} + \frac{9}{4}a,
\]
giving \(a = \frac{4}{5}\) and \(b = -\frac{1}{5}\). The second derivatives \(\frac{d^2V}{da^2}\) and \(\frac{d^2V}{db^2}\) are positive and so is the determinant of the Hessian matrix, i.e. \(\frac{d^2V}{da^2} \frac{d^2V}{db^2} - \left(\frac{d^2V}{da db}\right)^2\). Therefore the found solution is a minimum.


\[ V[Y] = C \left[ \int_0^\pi X(\omega) \, d\omega - \int_0^{2\pi} X(\omega) \, d\omega, \int_0^\pi X(\nu) \, d\nu - \int_0^{2\pi} X(\nu) \, d\nu \right] \]

\[ = \int_0^\pi \int_0^\pi r_X(\omega - \nu) \, d\omega \, d\nu - \int_0^\pi \int_0^{2\pi} r_X(\omega - \nu) \, d\omega \, d\nu \]

\[ - \int_0^{2\pi} \int_0^\pi r_X(\omega - \nu) \, d\omega \, d\nu + \int_0^\pi \int_0^{2\pi} r_X(\omega - \nu) \, d\omega \, d\nu, \]

where

\[ \int \int \cos^2\left(\frac{\omega - \nu}{2}\right) \, d\omega \, d\nu = \frac{1}{2} \int \int (1 + \cos(\omega - \nu)) \, d\omega \, d\nu, \]

\[ = \frac{1}{2} \int \int (1 + \cos(\omega) \cos(\nu) + \sin(\omega) \sin(\nu)) \, d\omega \, d\nu, \]

\[ = \frac{1}{2} \pi^2 + \frac{1}{2} \int \cos(\omega) \, d\omega \int \cos(\nu) \, d\nu + \frac{1}{2} \int \sin(\omega) \, d\omega \int \sin(\nu) \, d\nu. \]

The variance becomes

\[ V[Y] = \left( \frac{\pi^2}{2} + 2 \right) - \left( \frac{\pi^2}{2} - 2 \right) - \left( \frac{\pi^2}{2} - 2 \right) + \left( \frac{\pi^2}{2} + 2 \right) = 8. \]