Mathematical Statistics, Centre for Mathematical Sciences
Lund University
Solutions to exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2019-11-01

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Correct solutions of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits.The maximal number of credits are 90. The limit for passing the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) a) Correct answers: B) C) E) H)

b) A-D-I AR(1), B-F-G AR(2), C-E-H MA(2)

2) 3,4,5,7 are correct, 1,2,6,8,9,10 are false

3) a) Wrong. $f_s \geq 6$ i.e. $d \leq 1/6$.

b) Wrong. The resulting spectral density is

$$R_Y(f) = 1, \quad 1 \leq |f| \leq 2.$$

c) Correct.

4) a) The Wiener filter is given from

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)}.$$

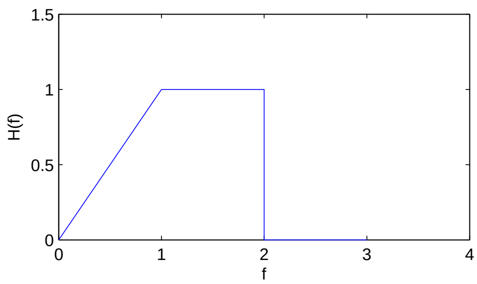
We find

$$R_S(f) + R_N(f) = \begin{cases} 1, & |f| \leq 1, \\ |f|, & 1 < |f| \leq 2, \end{cases}$$

and

$$\frac{R_S(f)}{R_S(f) + R_N(f)} = \begin{cases} |f|, & |f| \leq 1, \\ 1, & 1 < |f| \leq 2, \end{cases}$$

or according to the figure below.



The resulting SNR is

$$\text{SNR} = \frac{\int R_S(f)df}{\int \frac{R_S(f)R_N(f)}{R_S(f)+R_N(f)}df} = \frac{2\int_0^2 f df}{2\int_0^1 f(1-f)df} = \frac{4}{1/3} = 12.$$

The resulting $\text{SNR}_{opt} = 12$.

b) The SNR is defined as

$$\text{SNR} = \frac{E[S(t)^2]}{E[(Y(t) - S(t))^2]},$$

where

$$E[S^2(t)] = V[S(t)] = 2\int_0^2 R_S(f)df = 4,$$

and

$$\begin{aligned} E[(Y(t)-S(t))^2] &= E[(S_{out}(t)-S(t))^2]+E[N_{out}^2(t)], \\ &= V[(S_{out}(t)-S(t))]+V[N_{out}(t)]. \end{aligned}$$

where $Y(t)=S_{out}(t)+N_{out}(t)$, and the cross-covariance term is discarded as $S(t)$ and $N(t)$ are independent. We find

$$V[(S_{out}(t)-S(t))] = 2\int_0^2 |H(f)-1|^2 R_S(f)df = 2\int_0^1 R_S(f)df = 1,$$

and

$$V[N_{out}(t)] = 2\int_0^1 |H(f)|^2 R_N(f)df = 0.$$

Finally, the resulting

$$\text{SNR} = \frac{V[S(t)]}{V[(S_{out}(t)-S(t))]+V[N_{out}(t)]} = \frac{4}{1+0} = 4.$$

5) a) Define

$$\epsilon_{t+1} = X_{t+1}-Y_{t+1} = e_{t+1}+e_t+e_{t-1}-(e_t+e_{t-1}+e_{t-2}) = e_{t+1}-e_{t-2},$$

where $\epsilon_{t+1} \in N(0, V[e_{t+1}-e_{t-2}]) = N(0,1/2)$ as e_{t+1} and e_{t-2} are independent, and as linear combinations of Gaussian processes also are Gaussian processes. Then

$$P(|\epsilon_{t+1}|>1) = 2P(\epsilon_{t+1}>1) = 2(1-P(\epsilon_{t+1}\leq 1)) = 2(1-\Phi(\sqrt{2})) = 0.1586.$$

b) For a and b generally we get

$$\epsilon_{t+1} = e_{t+1}+e_t+e_{t-1}-a(e_t+e_{t-1}+e_{t-2})-b(e_{t-1}+e_{t-2}+e_{t-3}),$$

Minimizing $P(|\epsilon_{t+1}|>1)$ is the same as minimizing,

$$\begin{aligned} V[\epsilon_{t+1}] &= V[e_{t+1}+(1-a)e_t+(1-a-b)e_{t-1}-(a+b)e_{t-2}-be_{t-3}], \\ &= \frac{1}{4}(1+(1-a)^2+(1-a-b)^2+(a+b)^2+b^2), \\ &= \frac{1}{4}(1+(1-2a+a^2)+(1-2a-2b+2ab+a^2+b^2)+(a^2+2ab+b^2)+b^2). \end{aligned}$$

where $e_t \in N(0,1/4)$ are independent stochastic variables, or minimizing

$$V[\epsilon_{t+1}] = (1+a^2+b^2)r_X(0)+2a(b-1)r_X(1)-2br_X(2),$$

where $r_X(0)=\frac{3}{4}$, $r_X(1)=\frac{1}{2}$ and $r_X(2)=\frac{1}{4}$. Simplifications give

$$V[\epsilon_{t+1}] = \frac{1}{4}(3+3a^2+3b^2+4ab-4a-2b),$$

and the partial derivatives of $V[\epsilon_{t+1}]$ with respect to a and b are

$$\frac{dV}{da} = \frac{1}{4}(6a+4b-4) = \frac{3}{2}a+b-1=0,$$

and

$$\frac{dV}{db} = \frac{1}{4}(4a+6b-2) = a+\frac{3}{2}b-\frac{1}{2}=0.$$

The solution is found putting the first equation

$$b = 1-\frac{3}{2}a,$$

into the second equation

$$a = \frac{1}{2}-\frac{3}{2}(1-\frac{3}{2}a) = \frac{1}{2}-\frac{3}{2}+\frac{9}{4}a,$$

giving $a=\frac{4}{5}$ and $b=-\frac{1}{5}$. The second derivatives $\frac{d^2V}{da^2}$ and $\frac{d^2V}{db^2}$ are positive and so is the determinant of the Hessian matrix, i.e. $\frac{d^2V}{da^2}\frac{d^2V}{db^2}-(\frac{d^2V}{da\,db})^2$. Therefore the found solution is a minimum.

6)

$$\begin{aligned}
V[Y] &= C\left[\int_0^{\pi} X(\omega)\,d\omega - \int_{\pi}^{2\pi} X(\omega)\,d\omega, \int_0^{\pi} X(\nu)\,d\nu - \int_{\pi}^{2\pi} X(\nu)\,d\nu\right] \\
&= \int_0^{\pi}\int_0^{\pi} r_X(\omega-\nu)\,d\omega\,d\nu - \int_0^{\pi}\int_{\pi}^{2\pi} r_X(\omega-\nu)\,d\omega\,d\nu \\
&\quad - \int_{\pi}^{2\pi}\int_0^{\pi} r_X(\omega-\nu)\,d\omega\,d\nu + \int_{\pi}^{2\pi}\int_{\pi}^{2\pi} r_X(\omega-\nu)\,d\omega\,d\nu,
\end{aligned}$$

where

$$\begin{aligned}
\int\int \cos^2(\frac{\omega-\nu}{2})\,d\omega\,d\nu &= \frac{1}{2}\int\int(1+\cos(\omega-\nu))\,d\omega\,d\nu, \\
&= \frac{1}{2}\int\int(1+\cos(\omega)\cos(\nu)+\sin(\omega)\sin(\nu))\,d\omega\,d\nu, \\
&= \frac{1}{2}\pi^2 + \frac{1}{2}\int\cos(\omega)\,d\omega\int\cos(\nu)\,d\nu + \frac{1}{2}\int\sin(\omega)\,d\omega\int\sin(\nu)\,d\nu.
\end{aligned}$$

The variance becomes

$$V[Y] = (\frac{\pi^2}{2}+2) - (\frac{\pi^2}{2}-2) - (\frac{\pi^2}{2}-2) + (\frac{\pi^2}{2}+2) = 8.$$