Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2019-11-01
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

* *

Correct solutions of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits.The maximal number of credits are 90. The limit for passing the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

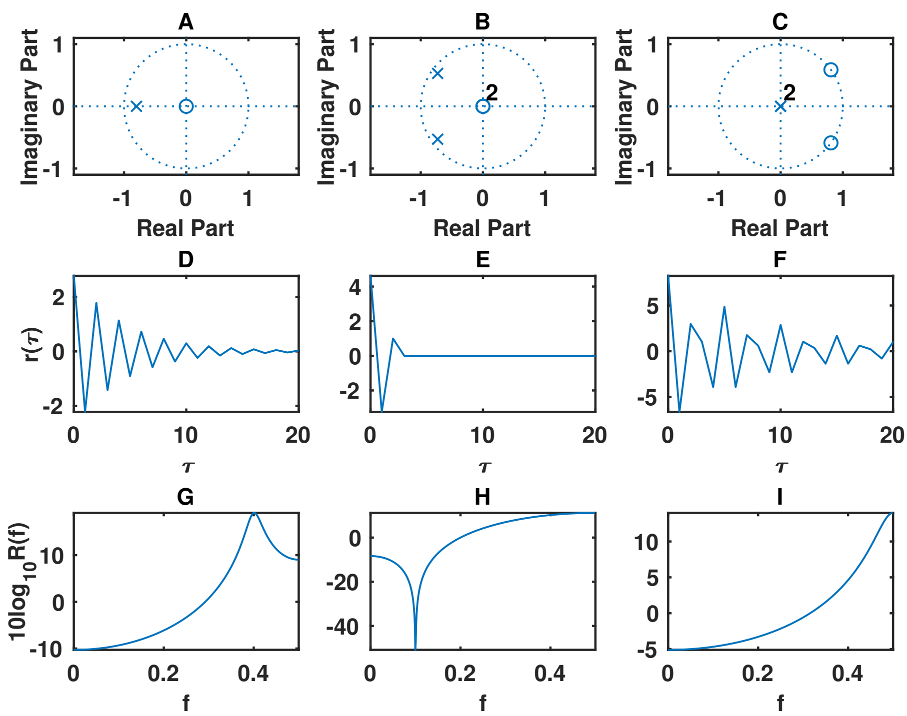
1) a) Identify which of the following statements A-H that are true for a real-valued discrete time weakly stationary process, where $r(\tau)$ is the covariance function and $R(f)$ is the spectral density. No motivation is needed.

A)	$r(0) < 0$	B)	$R(0) \geq 0$
C)	$\lvert r(-\tau)\rvert \leq r(0)$	D)	$R(-f) < 0$
E)	$r(\tau) = r(-\tau)$	F)	$r(\tau) > r(-\tau)$
G)	$R(f) = -R(-f)$	H)	$R(f) = \lvert R(f)\rvert$

(4p)

b) The figure shows the poles and zeros of 3 different discrete time stationary processes, the corresponding covariance functions and spectral densities in dB-scale. Combine the pole-zero plot with the corresponding covariance function and spectral estimate. Also state which type of process (AR, MA or ARMA) and their corresponding orders. No motivation is needed.

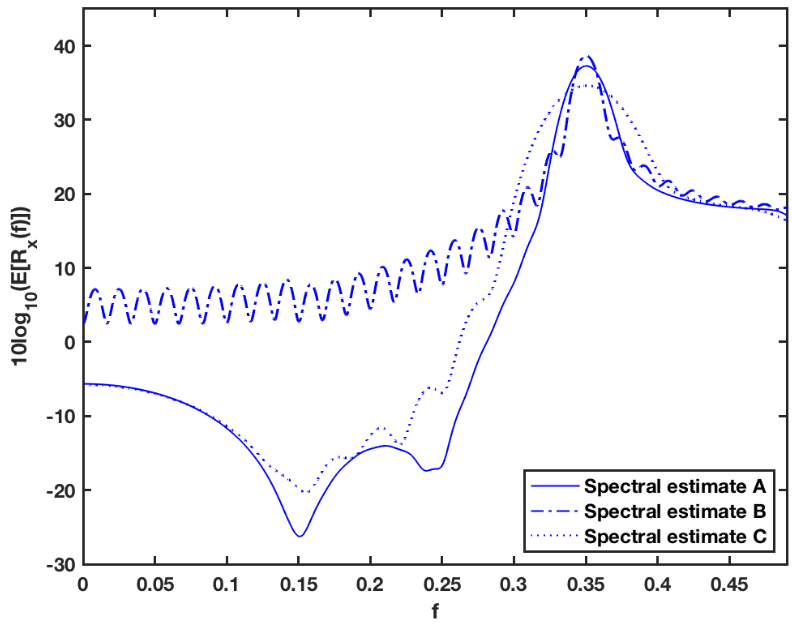
(6p)



2) The figure below shows the expected values of the spectral estimates using three methods: the periodogram with a rectangular window, the modified periodogram with a Hanning window and the Welch method using 3 Hanning windows with 50% overlap. The true process is an ARMA(2,4)-process and the number of data samples is $n = 60$. Determine if the following statements are correct. No motivation is needed.

1) Zero-padding will increase the frequency resolution of a spectral estimate.
2) Spectral estimate C is the periodogram with a rectangular window.
3) The periodogram with a rectangular window has the best resolution of the three methods.
4) Spectral estimate A is the modified periodogram with a Hanning window.
5) The modified periodogram with a Hanning window has less spectral leakage than the periodogram with a rectangular window.
6) The window lengths of the Hanning windows in the Welch method are $L = 20$.
7) The variance of the Welch method is 3 times smaller than the variance of the periodogram.
8) A spectral estimator with a wider main lobe can resolve two close peaks better than a spectral estimator with a more narrow main lobe.
9) The Welch method using 3 windows will have better resolution properties than the modified periodogram.
10) An increase of the number available samples i.e., $n \to \infty$, will decrease the variance of the periodogram.

(10p)



3) A continuous time zero-mean weakly stationary process $X(t)$, $t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \frac{|f|-1}{2}, \qquad 1 \le |f| \le 3.$$

The process is sampled into a discrete time process, $Y_t = X(t)$, $t = 0, \pm d, \pm 2d, \ldots$. Determine if the following statements are correct.

a) The largest possible sampling distance to avoid aliasing is $d = 0.2$. Motivate your answer.

b) After sampling with sampling frequency $f_s = 4$, the resulting spectral density is

$$R_Y(f) = 1 - |f|, \quad |f| \le 1.$$

Motivate your answer with drawings.

c) After sampling with sampling frequency $f_s = 2$, the resulting spectral density represents a discrete time stationary white noise sequence. Motivate your answer with drawings.

(10p)

4) A continuous time weakly stationary process $S(t), t \in \mathbb{R}$, with spectral density

$$R_S(f) = |f|, \quad |f| < 2,$$

is disturbed by a weakly stationary process, $N(t), t \in \mathbb{R}$, with spectral density

$$R_N(f) = 1 - |f|, \quad |f| < 1.$$

The processes $S(t)$ and $N(t)$ are both zero-mean and independent.

a) Calculate the Wiener filter and the corresponding SNR.

b) Calculate the SNR according to

$$\text{SNR} = \frac{E[S^2(t)]}{E[(Y(t) - S(t))^2]},$$

where $Y(t)$ now is the output from the filter

$$H(f) = 1, \quad 1 \le |f| \le 2,$$

with the input $S(t) + N(t)$.

All assumptions and calculations should be well motivated.

(20p)

5) In a simple model of the stockprice for the 'Mathematical statistics special fund II', the variation is modeled as an MA(2)-process

$$X_t = e_t + c_1 e_{t-1} + c_2 e_{t-2}, \quad t = 0, \pm 1, \pm 2, \dots$$

where $c_1 = c_2 = 1$ and the stationary Gaussian white noise sequence e_t has expected value $E[e_t] = 0$ and variance $V[e_t] = 1/4$. With use of

$$Y_{t+1} = aX_t + bX_{t-1},$$

the aim is to predict the variation at time $t+1$, i.e. to find an estimate of X_{t+1} with a small probability error.

a) Calculate $P(|X_{t+1} - Y_{t+1}| > 1)$ for the simple predictor $Y_{t+1} = X_t$, i.e. when $a = 1$ and $b = 0$ in the formula above.

b) Show that $a = 4/5$ and $b = -1/5$ minimize $P(|X_{t+1} - Y_{t+1}| > 1)$.

All assumptions and calculations should be well motivated.

(20p)

6) A metal band of circular shape should have the radius m. The deviation from this radius is modeled as a weakly stationary process, $X(\omega)$, $\omega \in \mathbb{R}$, where ω is the angle between a reference point and the angle to an actual point. The expected value is $E[X(\omega)] = 0$ and the covariance function is $r_X(\tau) = \cos^2(\tau/2)$. The variations in $X(\omega)$ cause the circular shape to be asymmetric and as a measure of the asymmetry we choose

$$Y = \int_0^{\pi} X(\omega)\, d\omega - \int_{\pi}^{2\pi} X(\omega)\, d\omega.$$

Compute the variance $V[Y]$.

All assumptions and calculations should be well motivated.

(20p)