Mathematical Statistics, Centre for Mathematical Sciences
Lund University
Solutions to exam in Stationary stochastic processes (FMSF10/MASC04)
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Solutions, correct and well motivated, of exercise 1-3 give 10 credits and of exercise 4-6 give 20 credits. Maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) a) The pole-zero plot in **A** has a zero at the unit circle at angle $\omega = 2\pi 0.2$ and a pole close to the unit circle at $\omega = 2\pi 0.3$. The corresponding spectral estimate should then have a zero at $f = 0.2$ and a peak at $f = 0.3$. It could possibly be figure **1** or **2**, but figure **1** seems to rather have two zeros and no peak. So most likely is that **A** belongs to **2**. The pole-zero plot in **B** has two zeros and is most likely to belong to figure **1**, where the pole-zero plot in **C**, that has a pole at $\omega = 2\pi 0.2$ belong to the spectral estimate in figure **3**, which has a peak at $f = 0.2$. Process **A-2** is an ARMA(2,2), **B-1** is an MA(4) and **C-3** an AR(2).

b) The periodogram variance is approximately $R_X^2(f)$ and the reduction of the Welch method is assumed to be the number of possible windows. The variance is approximately $V[\hat{R}_x(f)] = \frac{R_X^2(f)}{7}$ as $0.5 \cdot 64 \cdot 7 + 32 = 256$.

2) The covariance function for an MA(q)-process is,

$$r(\tau) = \sigma^2 \sum_{j-k=\tau} c_j c_k.$$

We get

$$r(\tau) = \begin{cases} 2(1^2 + (-1)^2) & = 4, \quad \tau = 0 \\ 2(1-1) & = 0, \quad |\tau| = 1 \\ 2(1-1) & = 0, \quad |\tau| = 2 \\ 2(-1) & = -2, \quad |\tau| = 3 \\ 0, & \quad |\tau| > 3. \end{cases}$$

The spectral density is

$$R_X(f) = 2|1 - e^{i2\pi 3f}|^2 = 4 - 4\cos(2\pi 3f) \quad -0.5 < f \le 0.5.$$

3) The expected value of the derivative is always $E[X'(t)] = 0$ and the spectral density is $R_{X'}(f) = (2\pi f)^2 R_X(f)$. The variance of the derivative is

$$\begin{aligned} V[X'(t)] &= \int R_{X'}(f)df = c\int_{f_0}^{f_0} (2\pi f)^2(1 - |f|/f_0)df \\ &= 2c\int_0^{f_0} (2\pi f)^2(1 - f/f_0)df = 8\pi^2 c f_0^3/12 = 1. \end{aligned}$$

4) a) The Yule-Walker-equations give

$$\begin{aligned} 4 + 2a_1 - 0.5a_2 &= \sigma^2 \\ 2 + 4a_1 + 2a_2 &= 0 \\ -0.5 + 2a_1 + 4a_2 &= 0, \end{aligned}$$

with the solution $a_1 = -0.75$, $a_2 = 0.5$ and $\sigma^2 = 2.25$. The spectral density function is

$$R_x(f) = \frac{2.25}{|1 - 0.75e^{-i2\pi f} + 0.5e^{-i4\pi f}|^2} =$$

$$\frac{2.25}{1.8125 - 2.25\cos(2\pi f) + \cos(4\pi f)}.$$

b) The spectral density of the noise disturbance is $R_w(f) = 1$. The Wiener filter becomes

$$\begin{aligned} H(f) &= \frac{R_x(f)}{R_x(f) + R_w(f)} \\ &= \frac{2.25}{2.25 + |1 - 0.75e^{-i2\pi f} + 0.5e^{-i4\pi f}|^2} \\ &= \frac{2.25}{4.0625 - 2.25\cos(2\pi f) + \cos(4\pi f)}. \end{aligned}$$

5) a) **Correct:** The frequencies after sampling are 1000 Hz (no aliasing), $5000 - f_s = -2000$ giving 2000 Hz, (aliasing of $f_2 = 5000$ Hz), and $17000 - 2 \cdot f_s = 3000$ Hz, (aliasing of $f_3 = 17000$ Hz).

b) **False:** The frequency function is

$$H(f) = \begin{cases} 1 - \frac{1}{10000} \cdot |f| & |f| \leq 10000, \\ 0 & |f| > 10000. \end{cases}$$

The remaining frequencies after the filtering are 1000 Hz and 5000 Hz where the frequency 17000 Hz is cancelled. The process $Y(t)$ is sampled with $f_s = 7000$ Hz and the resulting frequencies are 1000 Hz (no aliasing), $5000 - f_s = -2000$ giving 2000 Hz, (aliasing of $f_2 = 5000$ Hz).

c) **Correct:** The frequencies after sampling with $f_s = 20000$ Hz are 1000 Hz (no aliasing), 5000 Hz (no aliasing) and $17000 - f_s = -3000$ giving 3000 Hz (aliasing of $f_3 = 17000$ Hz). The frequencies of the corresponding discrete time process Z_t are $1000/f_s = 1/20$, $3000/f_s = 3/20$ and $5000/f_s = 1/4$. Two frequencies remain, 1/20 and 3/20, after filtering with the discrete time lowpass filter which passes all frequencies below $|\nu| \leq 3/16$.

6) The output spectral density is

$$R_Y(f) = \frac{R_X(f)}{|1 + 0.5e^{-i2\pi f}|^2},$$

with $H(f) = \frac{1}{1+0.5e^{-i2\pi f}} = \sum_{k=0}^{\infty}(-0.5)^k e^{-i2\pi f k}$, i.e., the impulse response is $h(k) = (-0.5)^k$, $k \geq 0$. We can then write $Y_k = \sum_{u=0}^{\infty}(-0.5)^u X_{k-u}$. The cross-covariance is

$$r_{X,Y}(\tau) = C[X_k, \sum_{u=0}^{\infty}(-0.5)^u X_{k-u+\tau}] = \sum_{u=0}^{\infty}(-0.5)^u C[X_k, X_{k-u+\tau}],$$

yielding

$$r_{X,Y}(\tau) = r_X(\tau) - 0.5 r_X(\tau - 1) + 0.25 r_X(\tau - 2) + \ldots.$$

We obtain $r_{X,Y}(\tau) = 0$ for $\tau \leq -2$, $r_{X,Y}(-1) = r_X(-1) = 0.2$, $r_{X,Y}(0) = r_X(0) - 0.5 r_X(-1) = 1 - 0.1 = 0.9$, and

$$r_{X,Y}(\tau) = (-0.5)^{\tau+1} r_X(-1) + (-0.5)^{\tau} r_X(0) + (-0.5)^{\tau-1} r_X(1),$$

when $\tau \geq 1$, implying

$$r_{X,Y}(\tau) = (-0.5)^{\tau}(-0.1 + 1 - 0.4) = 0.5(-0.5)^{\tau}.$$