Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2019-01-07
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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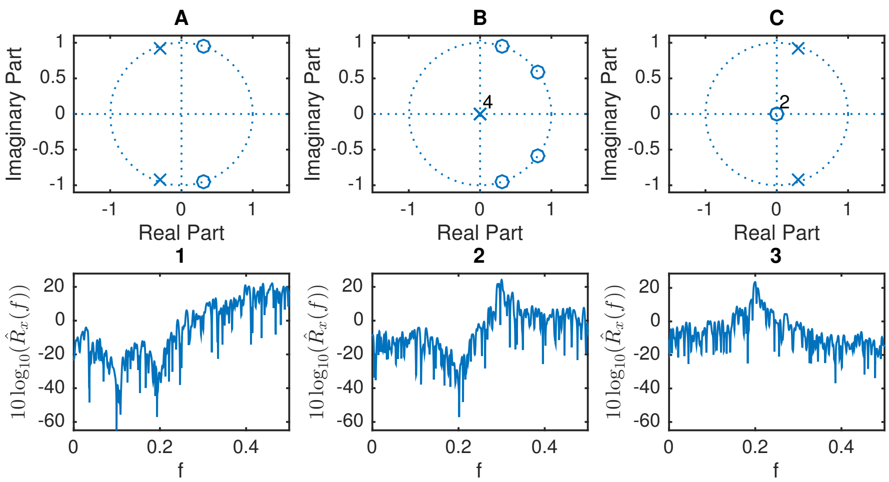
Solutions, correct and well motivated, of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits. The maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) a) The figure shows the poles and zeros of 3 different discrete-time stationary stochastic processes and the corresponding modified periodogram estimates using a Hanning window, in dB-scale. Combine the pole-zero plot with the corresponding spectral estimate. Motivate your answer. Also state which type of process (AR, MA or ARMA) and the corresponding orders.

(6p)



b) We have access to 256 samples of data, where the true spectral density function is $R_X(f)$. The Welch method, with window length 64 in each periodogram and 50% overlap between windows, is applied. Approximately how large is the variance for $0 < |f| < 0.5$? Motivate your answer.

(4p)

2) Determine the covariance function and the spectral density of the MA(3)-process

$$X_t = e_t - e_{t-3},$$

where the zero-mean Gaussian white noise process e_t, $t = 0, \pm 1, \pm 2, \ldots$, has the variance $V[e_t] = 2$.

(10p)

3) The differentiable stationary process $X(t)$, $t \in \mathbb{R}$, has the spectral density

$$R_X(f) = c(1 - |f|/f_0), \quad |f| < f_0,$$

where $f_0 = 1/c = \sqrt{6}/2\pi$. Compute the expected value and the variance of the derivative $X'(t)$, $t \in \mathbb{R}$.

(10p)

4) a) For an AR(2)-process X_t, $t = 0, \pm 1, \pm 2, \ldots$, the following covariance function values are given,

$$r_X(\tau) = \begin{cases} 4 & \tau = 0, \\ 2 & |\tau| = 1, \\ -0.5 & |\tau| = 2. \end{cases}$$

Determine the spectral density of the AR(2)-process.

(10p)

b) Compute the frequency function of a Wiener filter which reconstructs the AR(2)-process X_t in exercise 4a) from the measurement data $Y_t = X_t + W_t$, where W_t, $t = 0, \pm 1, \pm 2, \ldots$, is Gaussian white noise with expected value 0 and variance 1. For full number of credits, the filter should be simplified as far as possible and presented in a real-valued form.

(10p)

5) A stationary continuous time process is given as

$$X(t) = \sum_{k=1}^{3} A_k \cos(2\pi f_k t + \phi_k), \quad -\infty < t < \infty,$$

where the stochastic variables A_k, ϕ_k, $k = 1 \ldots 3$, all are independent and $\phi_k \in \text{Rect}(0, 2\pi)$. The frequencies are $f_1 = 1000$ Hz, $f_2 = 5000$ Hz, and $f_3 = 17000$ Hz.

Determine if the following statements are correct or false. **Note: for full number of credits the answer must include an extensive motivation.**

a) The process $X(t)$ is sampled with $f_s = 7000$ Hz. The frequencies after sampling are 1000 Hz, 2000 Hz, and 3000 Hz.

(4p)

b) The process $X(t)$ is filtered giving $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$, with

$$h(t) = \begin{cases} 10000 & \text{if } t = 0, \\ \frac{1}{20000\,\pi^2 t^2}(1 - \cos(20000\pi t)) & \text{if } t \neq 0. \end{cases}$$

Then the process $Y(t)$ is sampled with $f_s = 7000$ Hz. The resulting frequencies after sampling are 1000 Hz and 3000 Hz.

(8p)

c) The process $X(t)$ is sampled with $f_s = 20000$ Hz and is then converted into the discrete time sequence Z_t, $t = 0, \pm1, \pm2, \ldots$. The sequence Z_t is filtered giving $W_t = \sum_{u=-\infty}^{\infty} g(u) Z_{t-u}$ using the ideal low-pass filter $g(t)$, $t = 0, \pm1, \pm2, \ldots$, with corresponding frequency function,

$$G(\nu) = \begin{cases} 1 & |\nu| \leq \frac{3}{16}, \\ 0 & \frac{3}{16} < |\nu| \leq \frac{1}{2}, \end{cases}$$

where ν is normalized frequency. The sequence W_t, $t = 0, \pm1, \pm2, \ldots$, contains only two frequencies after the discrete time filtering.

(8p)

6) The weakly stationary processes X_t, $t = 0, \pm1, \pm2, \ldots$ and Y_t, $t = 0, \pm1, \pm2, \ldots$ are input and output of a linear filter according to

$$Y_t + 0.5 Y_{t-1} = X_t, \quad \text{for } t = 0, \pm1, \pm2, \ldots.$$

The process X_t has the covariance function $r_X(0) = 1$, $r_X(\pm1) = 0.2$, and zero for all other values. Determine the cross-covariance function

$$r_{X,Y}(\tau) = C[X_t, Y_{t+\tau}]$$

for all values of τ.

(20p)