1) a) The figure shows the poles and zeros of 3 different discrete-time stationary stochastic processes and the corresponding modified periodogram estimates using a Hanning window, in dB-scale. Combine the pole-zero plot with the corresponding spectral estimate. Motivate your answer. Also state which type of process (AR, MA or ARMA) and the corresponding orders.

![Diagram of pole-zero plots and periodograms](image)

b) We have access to 256 samples of data, where the true spectral density function is $R_X(f)$. The Welch method, with window length 64 in each periodogram and 50% overlap between windows, is applied. Approximately how large is the variance for $0 < |f| < 0.5$? Motivate your answer.

2) Determine the covariance function and the spectral density of the MA(3)-process

$$X_t = e_t - e_{t-3},$$

where the zero-mean Gaussian white noise process $e_t$, $t = 0, \pm 1, \pm 2, \ldots$, has the variance $V[e_t] = 2$. 

(10p)
3) The differentiable stationary process $X(t), t \in \mathbb{R}$, has the spectral density

$$R_X(f) = c(1 - |f|/f_0), \quad |f| < f_0,$$

where $f_0 = 1/c = \sqrt{6}/2\pi$. Compute the expected value and the variance of the derivative $X'(t), t \in \mathbb{R}$.

(10p)

4) a) For an AR(2)-process $X_t, t = 0, \pm 1, \pm 2, \ldots$, the following covariance function values are given,

$$r_X(\tau) = \begin{cases} 4 & \tau = 0, \\ 2 & |\tau| = 1, \\ -0.5 & |\tau| = 2. \end{cases}$$

Determine the spectral density of the AR(2)-process.

(10p)

b) Compute the frequency function of a Wiener filter which reconstructs the AR(2)-process $X_t$ in exercise 4a) from the measurement data $Y_t = X_t + W_t$, where $W_t, t = 0, \pm 1, \pm 2, \ldots$, is Gaussian white noise with expected value 0 and variance 1. For full number of credits, the filter should be simplified as far as possible and presented in a real-valued form.

(10p)

5) A stationary continuous time process is given as

$$X(t) = \sum_{k=1}^{3} A_k \cos(2\pi f_k t + \phi_k), \quad -\infty < t < \infty,$$

where the stochastic variables $A_k, \phi_k, k = 1 \ldots 3$, all are independent and $\phi_k \in \text{Rect}(0, 2\pi)$. The frequencies are $f_1 = 1000$ Hz, $f_2 = 5000$ Hz, and $f_3 = 17000$ Hz.

Determine if the following statements are correct or false. **Note: for full number of credits the answer must include an extensive motivation.**

a) The process $X(t)$ is sampled with $f_s = 7000$ Hz. The frequencies after sampling are 1000 Hz, 2000 Hz, and 3000 Hz.

(4p)

b) The process $X(t)$ is filtered giving $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$, with

$$h(t) = \begin{cases} 10000 & \text{if } t = 0, \\ \frac{1}{20000 \pi^2 t^2}(1 - \cos(20000 \pi t)) & \text{if } t \neq 0. \end{cases}$$

Then the process $Y(t)$ is sampled with $f_s = 7000$ Hz. The resulting frequencies after sampling are 1000 Hz and 3000 Hz.

(8p)
c) The process \( X(t) \) is sampled with \( f_s = 20000 \) Hz and is then converted into the discrete time sequence \( Z_t, t = 0, \pm 1, \pm 2, \ldots \). The sequence \( Z_t \) is filtered giving \( W_t = \sum_{u=-\infty}^{\infty} g(u)Z_{t-u} \) using the ideal low-pass filter \( g(t), t = 0, \pm 1, \pm 2, \ldots \), with corresponding frequency function, 

\[
G(\nu) = \begin{cases} 
1 & |\nu| \leq \frac{3}{16}, \\
0 & \frac{3}{16} < |\nu| \leq \frac{1}{2},
\end{cases}
\]

where \( \nu \) is normalized frequency. The sequence \( W_t, t = 0, \pm 1, \pm 2, \ldots \), contains only two frequencies after the discrete time filtering.

\[ \text{(8p)} \]

6) The weakly stationary processes \( X_t, t = 0, \pm 1, \pm 2, \ldots \) and \( Y_t, t = 0, \pm 1, \pm 2, \ldots \) are input and output of a linear filter according to

\[
Y_t + 0.5Y_{t-1} = X_t, \text{ for } t = 0, \pm 1, \pm 2, \ldots
\]

The process \( X_t \) has the covariance function \( r_X(0) = 1, r_X(\pm 1) = 0.2 \), and zero for all other values. Determine the cross-covariance function

\[
r_{X,Y}(\tau) = C[X_t, Y_{t+\tau}]
\]

for all values of \( \tau \).

\[ \text{(20p)} \]