Mathematical Statistics, Centre for Mathematical Sciences
Lund University
Solutions to exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-11-02

* *

Solutions, correct and well motivated, of exercise 1-3 give 10 credits and of exercise 4-6 give 20 credits. Maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) The processes must be time-discrete as they are either AR or MA. A,C,H are spectral densities as they are positive and have a x-scale 0-0.5. D,G,I are covariance functions as they are the only ones with the largest values at $t = 0$. B,E,F are realizations. B has a slow variation corresponding to covariance function G and spectral density A. A faster variation is found for E with covariance function I and spectral density C. F is a more noise process (more frequencies) which corresponds to the spectral density H. The covariance function in D (MA-process, with few values not equal to zero) can easily be transformed to a spectral density consisting of cosine functions. The covariance functions I and G slowly goes to zero and should accordingly be AR-processes.

2) Investigate the new process $Y(t) = X'(t) - X(t)$ and $P(Y(t) \geq 0)$, where

$$E[Y(t)] = 0 - 2 = -2.$$

and

$$V[Y(t)] = C[X'(t) - X(t), X'(t) - X(t)] = V[X'(t)] + V[X(t)] = -r''_X(0) + r_X(0),$$

as the cross-covariance at $\tau = 0$ is zero. With $r'_X(\tau) = -2\tau e^{-\tau^2}$, and $r''_X(\tau) = -2e^{-\tau^2} + 4\tau^2 e^{-\tau^2}$, the variance becomes $V[Y(t)] = 2 + 1 = 3$. The probability

$$P(Y(t) \geq 0) = 1 - \Phi(\frac{0-(-2)}{\sqrt{3}}) = 1 - \Phi(1.15) = 1 - 0.875 = 0.125.$$

3) a) Wrong, the Welch method has shorter windows and thereby the resolution is worse than the periodogram.

b) Correct, as we can see that the spectrum estimate level at $f > 0.3$ for A is much higher than in B.

c) Wrong, the overlap is 50% so the window lengths are $L = 40$.

d) Correct, the variance is reduced a factor K where K is the number of windows. The standard deviation is then reduced a factor $\sqrt{K} = \sqrt{4} = 2$.

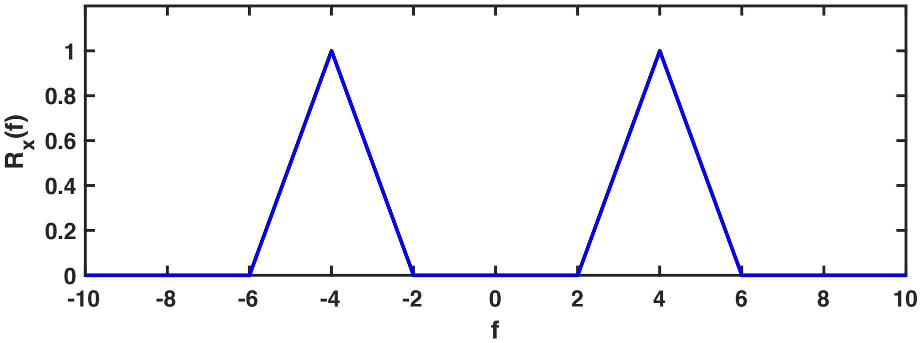
e) Wrong, as the resolution of peaks are worse in B and the leakage and bias at high frequencies is worse in A.

4) Yule-Walker gives

$$\begin{aligned} r_X(0) + a_1 \cdot 1.6 + a_2 \cdot 0.40 &= \sigma^2 \\ 1.6 + a_1 \cdot r_X(0) + a_2 \cdot 1.6 &= 0 \\ 0.4 + a_1 \cdot 1.6 + a_2 \cdot r_X(0) &= 0 \\ -0.4 + a_1 \cdot 0.4 + a_2 \cdot 1.6 &= 0 \end{aligned}$$

solved by $a_2 = 1/4 - 1/4a_1$, $r_X(0) = -\frac{1+a_2}{a_1} \cdot 1.6$ and $a_1^2 + 2/3 \cdot a_1 - 1/3 = 0$ with two $a_1 = -1$ and $a_1 = 1/3$. We have the two following alternatives: Alternative 1 with $a_1 = 1/3$, $a_2 = 1/6$ gives $r_X(0) = -5.6$ which is not a variance. Alternative 2 with $a_1 = -1$, $a_2 = 1/2$ gives $r_X(0) = 2.4$ and $\sigma^2 = 1$ which is the solution.

5) a) The spectral density is found as,



and as the covariance function of one triangular function located with centre $f = 0$, i.e $R_{x0}(f) = (1 - 0.5|f|)$, $f \leq 2$, is

$$r_{x_0}(\tau) = \frac{1}{(2\pi\tau)^2}\left(1 - \cos(4\pi\tau)\right), \quad \tau \neq 0,$$

and $r_{x_0}(0) = 2$, the frequency-shifted covariance function becomes

$$\begin{aligned} r_x(\tau) &= \frac{1}{(2\pi\tau)^2}\left(1 - \cos(4\pi\tau)\right) \cdot (e^{i2\pi 4\tau} + e^{-i2\pi 4\tau}), \quad \tau \neq 0, \\ &= \frac{1}{2\pi^2\tau^2}\left(1 - \cos(4\pi\tau)\right) \cdot \cos(8\pi\tau) \quad \tau \neq 0, \end{aligned}$$

and $r_x(0) = 4$.

b) The sampling frequency $f_s = 2$ will cause aliasing and the resulting spectral density will be constant with $R_Z(f) = 2$, $-1 < f \leq 1$.

c) We find $r_Z(0) = 4$ and $r_Z(\tau) = 0$, otherwise. Then

$$V[\widehat{m}_n] = \frac{nV[Z_t]}{n^2} = \frac{4}{n}.$$

6) The variance is

$$\begin{aligned} V[U] &= V[\frac{1}{3}(X_0 + X_1 + X_2 + Y_1 + Y_2)] \\ &= \frac{1}{9}(V[X_0 + X_1 + X_2] + V[Y_1] + V[Y_2]) \\ &= \frac{1}{9}(C[X_0 + X_1 + X_2, X_0 + X_1 + X_2] + V[Y_1] + V[Y_2]) \\ &= \frac{1}{9}(3r_X(0) + 2r_X(a) + 2r_X(a+b) + 2r_X(b) + a\sigma^2 + b\sigma^2) \\ &= \frac{1}{9}(3 + 2e^{-|a|} + 2e^{-|a+b|} + 2e^{-|b|} + (a+b)0.01) \\ &= \frac{1}{9}(3 + 2e^{-a} + 2e^{-(a+b)} + 2e^{-b} + (a+b)0.01). \end{aligned}$$

The derivatives with respect to a and b are

$$\begin{aligned} \frac{\partial f}{\partial a} &= \frac{1}{9}(-2e^{-a} - 2e^{-a-b} + 0.01) = 0 \\ \frac{\partial f}{\partial b} &= \frac{1}{9}(-2e^{-b} - 2e^{-a-b} + 0.01) = 0, \end{aligned}$$

where $a = b$. With $x = e^{-a}$, the positive solution is given from $x^2 + x - 0.005 = 0$, as $x = 4.975 \cdot 10^{-3}$ which gives $a = b = 5.3$.