Solutions to exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-11-02

Solutions, correct and well motivated, of exercise 1-3 give 10 credits and of exercise 4-6 give 20 credits. Maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) The processes must be time-discrete as they are either AR or MA. A,C,H are spectral densities as they are positive and have a x-scale 0-0.5. D,G,I are covariance functions as they are the only ones with the largest values at \( t = 0 \). B,E,F are realizations. B has a slow variation corresponding to covariance function G and spectral density A. A faster variation is found for E (MA-process, with few values not equal to zero) can easily be transformed to a spectral density consisting of cosine functions. The covariance functions I and G slowly goes to zero and should accordingly be AR-processes.

2) Investigate the new process \( Y(t) = X'(t) - X(t) \) and \( P(Y(t) \geq 0) \), where

\[
E[Y(t)] = 0 - 2 = -2.
\]

and

\[
V[Y(t)] = C[X'(t) - X(t), X'(t) - X(t)] = V[X'(t)] + V[X(t)] = -r_X''(0) + r_X(0),
\]

as the cross-covariance at \( \tau = 0 \) is zero. With \( r_X'(\tau) = -2\tau e^{-\tau^2} \), and \( r_X''(\tau) = -2e^{-\tau^2} + 4\tau^2 e^{-\tau^2} \), the variance becomes \( V[Y(t)] = 2 + 1 = 3 \). The probability

\[
P(Y(t) \geq 0) = 1 - \Phi\left(\frac{0 - (-2)}{\sqrt{3}}\right) = 1 - \Phi(1.15) = 1 - 0.875 = 0.125.
\]

3) a) Wrong, the Welch method has shorter windows and thereby the resolution is worse than the periodogram.

b) Correct, as we can see that the spectrum estimate level at \( f > 0.3 \) for A is much higher than in B.

c) Wrong, the overlap is 50% so the window lengths are \( L = 40 \).

d) Correct, the variance is reduced a factor \( K \) where \( K \) is the number of windows. The standard deviation is then reduced a factor \( \sqrt{K} = \sqrt{4} = 2 \).

e) Wrong, as the resolution of peaks are worse in B and the leakage and bias at high frequencies is worse in A.

4) Yule-Walker gives

\[
\begin{align*}
r_X(0) + a_1 \cdot 1.6 + a_2 \cdot 0.40 & = \sigma^2 \\
1.6 + a_1 \cdot r_X(0) + a_2 \cdot 1.6 & = 0 \\
0.4 + a_1 \cdot 1.6 + a_2 \cdot r_X(0) & = 0 \\
-0.4 + a_1 \cdot 0.4 + a_2 \cdot 1.6 & = 0
\end{align*}
\]

solved by \( a_2 = 1/4 - 1/4a_1 \), \( r_X(0) = -\frac{1 + a_2 a_1}{a_1^2} \cdot 1.6 \) and \( a_1^2 + 2/3 \cdot a_1 - 1/3 = 0 \) with two \( a_1 = -1 \) and \( a_1 = 1/3 \). We have the two following alternatives: Alternative 1 with \( a_1 = 1/3 \), \( a_2 = 1/6 \) gives \( r_X(0) = -5.6 \) which is not a variance. Alternative 2 with \( a_1 = -1 \), \( a_2 = 1/2 \) gives \( r_X(0) = 2.4 \) and \( \sigma^2 = 1 \) which is the solution.
a) The spectral density is found as,

\[
R_x(f) = 1 - 10^{-8} - 6 - 4 - 2 0 2 4 6 8 10
\]

and as the covariance function of one triangular function located with centre \( f = 0 \), i.e
\[
R_{x_0}(f) = (1 - 0.5|f|), \quad f \leq 2,
\]
is
\[
r_{x_0}(\tau) = \frac{1}{(2\pi\tau)^2} (1 - \cos(4\pi\tau)), \quad \tau \neq 0,
\]
and \( r_{x_0}(0) = 2 \), the frequency-shifted covariance function becomes
\[
r_x(\tau) = \frac{1}{(2\pi\tau)^2} (1 - \cos(4\pi\tau)) \cdot (e^{i2\pi\tau} + e^{-i2\pi\tau}), \quad \tau \neq 0,
\]
and \( r_x(0) = 4 \).

b) The sampling frequency \( f_s = 2 \) will cause aliasing and the resulting spectral density will be constant with \( R_Z(f) = 2, -1 < f \leq 1 \).

c) We find \( r_Z(0) = 4 \) and \( r_Z(\tau) = 0 \), otherwise. Then
\[
V[\tilde{m}_{n}] = \frac{nV[Z_t]}{n^2} = \frac{4}{n}.
\]

6) The variance is
\[
V[U] = V[\frac{1}{3}(X_0 + X_1 + X_2 + Y_1 + Y_2)]
\]
\[
= \frac{1}{9}(V[X_0 + X_1 + X_2] + V[Y_1] + V[Y_2])
\]
\[
= \frac{1}{9}(C[X_0 + X_1 + X_2, X_0 + X_1 + X_2] + V[Y_1] + V[Y_2])
\]
\[
= \frac{1}{9}(3r_X(0) + 2r_X(a) + 2r_X(a + b) + 2r_X(b) + a\sigma^2 + b\sigma^2)
\]
\[
= \frac{1}{9}(3 + 2e^{-a} + 2e^{-(a+b)} + 2e^{-b} + (a + b)0.01)
\]
\[
= \frac{1}{9}(3 + 2e^{-a} + 2e^{-(a+b)} + 2e^{-b} + (a + b)0.01).
\]

The derivatives with respect to \( a \) and \( b \) are
\[
\frac{\partial f}{\partial a} = \frac{1}{9}(-2e^{-a} - 2e^{-a-b} + 0.01) = 0
\]
\[
\frac{\partial f}{\partial b} = \frac{1}{9}(-2e^{-b} - 2e^{-a-b} + 0.01) = 0,
\]
where \( a = b \). With \( x = e^{-a} \), the positive solution is given from \( x^2 + x - 0.005 = 0 \), as \( x = 4.975 \cdot 10^{-3} \) which gives \( a = b = 5.3 \).