Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-11-02
Time: 14.00 – 19.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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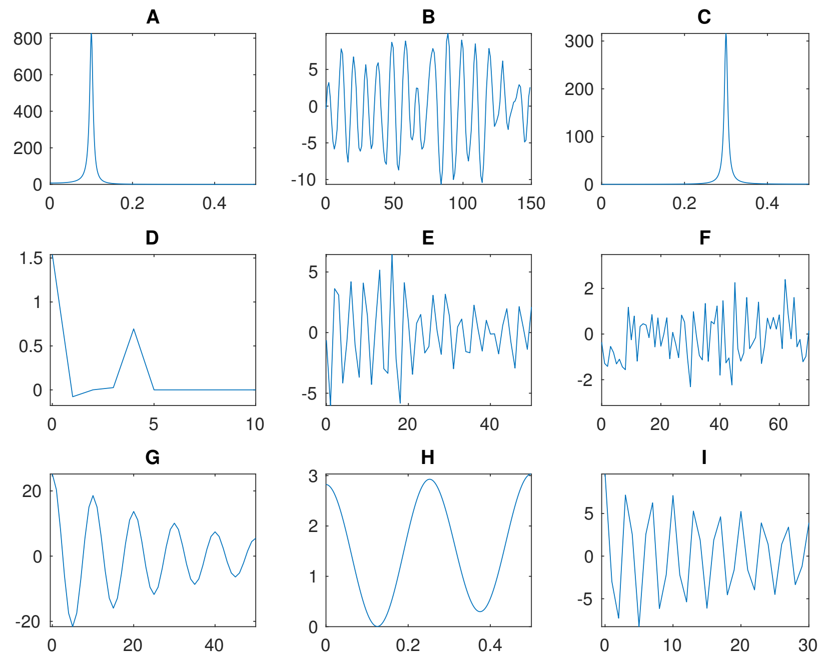
Solutions, correct and well motivated, of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits. The maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) The following figures show realizations, covariance functions and spectral densities of three different discrete time stationary processes. Determine, with a motivation, what figures that are realizations, covariance functions and spectral densities. Also state which figures that belong to each of the processes. Finally decide of which type and order each of the three different processes are, autoregressive (AR) or moving average (MA). The order can be assumed to be smaller than seven. Important! To receive full number of credits correct motivations have to be given for all answers.

(10p)



2) A stationary Gaussian process $X(t)$, $t \in \mathbb{R}$, has expected value $E[X(t)] = 2$ and covariance function

$$r_X(\tau) = e^{-\tau^2}.$$

Compute the probability,

$$P(X'(t) \geq X(t)).$$

(10p)

3) The figures below show the expected values of the spectrum estimates from two methods, the periodogram and the Welch method. The true process is an AR(6)-process and the number of data samples is $n = 100$. The periodogram uses a rectangle window and the Welch method applies 4 Hanning windows with 50 % overlap. Determine, with a short motivation, if the following statements are correct.

a) Using the Welch method gives a better resolution of close peaks.

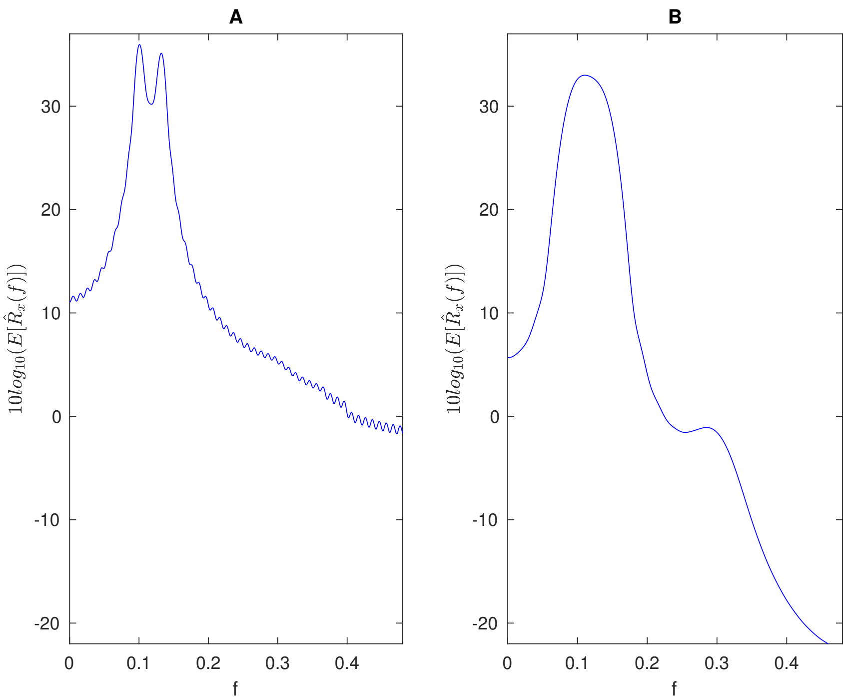
b) The high sidelobes of the periodogram causes leakage and severe bias for frequencies $f > 0.2$.

c) The window lengths of the different Hanning windows of the Welch method described above is $L = 30$ (total data length $n = 100$).

d) The standard deviation of the Welch spectrum estimate is reduced a factor 2 compared to the standard deviation of the periodogram.

e) Figure A shows the Welch method and figure B the periodogram.

(10p)



4) An AR(2)-process

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} = e_t$$

where $e_t \in N(0, \sigma^2)$, has

$$\begin{aligned} r_x(1) &= 1.6 \\ r_x(2) &= 0.4 \\ r_x(3) &= -0.4. \end{aligned}$$

Determine the parameters a_k, $k = 1 \ldots p$, of the AR-process.

(20p)

5) We define the stationary Gaussian process $Y(t) = m + X(t)$, $t \in \mathbb{R}$, where m is an unknown constant. The process $X(t)$ is assumed to have $E[X(t)] = 0$ and the spectral density

$$R_X(f) = \begin{cases} \left(1 - |\frac{f}{2} - 2|\right) & \text{for} \quad |f-4| \le 2, \\ \left(1 - |\frac{f}{2} + 2|\right) & \text{for} \quad |f+4| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

An estimate of m should be found as

$$\widehat{m}_n = \frac{Y(d) + Y(2d) + \ldots Y(nd)}{n},$$

by sampling the process $Y(t)$ with sample distance d.

a) Calculate the covariance function $r_Y(\tau)$.

(8p)

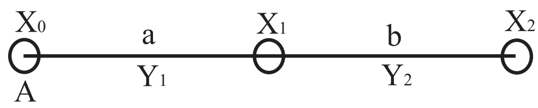
b) Determine the spectral density of the sampled process $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$ for the sample distance $d = 1/2$.

(8p)

c) Determine the variance, $V[\widehat{m}_n]$, when $d = 1/2$.

(4p)

6) A weak radiation field should be measured with three detectors placed in a straight line with the distances a and b according to figure below, where the registration is made at point **A**:



The strength of the field is $X(d)$, where d is the distance from the point **A** in the figure where the registration is made, is assumed to be a stationary process with the unknown average m. The covariance function for the field between two points with distance τ is $r_X(\tau) = e^{-|\tau|}$. The average level m should be estimated with the best possible precision. The detectors are connected with **A** by a wire, which causes disturbances in the transmission.

At point **A**

$$U = \frac{1}{3}(X_0 + X_1 + X_2 + Y_1 + Y_2),$$

is registered where $X_0 = X(0)$, $X_1 = X(a)$, $X_2 = X(a+b)$, and the disturbances Y_1 and Y_2 are independent stochastic variables with expected value 0 and variances that are proportional to the length of the transmission wires, i.e., $V[Y_1] = a\sigma^2$, $V[Y_2] = b\sigma^2$, with $\sigma^2 = 0.01$. How far apart should the detectors be placed if $V[U]$ should be as small as possible?

(20p)