1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are B and F. A spectral density is always positive, which is true for C and E. Thus A and D are realizations.

D varies faster than A, and D then corresponds to the strong spectral peak at $f = 0.25$ in C, where A corresponds to the more low-frequency spectral peak in E. The same difference is seen for the covariances B and F where F is the one with the slowest oscillation. Therefore A, F and E belong together and D, B and C.

The covariance function in B is the MA(6)-process as it is zero for $\tau > 6$. The covariance function in F belongs to the AR(2)-process, as the response certainly has covariance values for $\tau > 6$ and cannot be the MA(6)-process.

2) The spectral density for the input process is

$$R_X(f) = \pi e^{-2\pi |f|},$$

and for the output process

$$R_Y(f) = |H(f)|^2 R_X(f) = \pi e^{-2\pi |f|}, \quad 1 < |f| < 2.$$ 

The variance is

$$V[Y(t)] = \int_{-\infty}^{\infty} R_Y(f) df = 2 \int_1^2 \pi e^{-2\pi |f|} df = e^{-2\pi} - e^{-4\pi}.$$

3) a) The sampling interval $d$ is decided from $\frac{1}{2d} > 1.5$, giving $d < \frac{1}{3}$.

b) Let $Z_t$ denote the sampled process for $d = 1$ with spectral density

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f + k).$$

Aliasing will occur and for $0 \leq f \leq \frac{1}{2}$ we have submissions from $k = 0$ and $k = \pm 1$ as

$$R_Z(f) = R_X(f) + R_X(f + 1) + R_X(f - 1) = 1.5 - f + 1.5 - |f + 1| + 1.5 - |f - 1| = 1.5 - f + 1.5 - f - 1 + 1.5 + f - 1 = 2.5 - f.$$

This gives $R_Z(f) = 2.5 - |f|$, for $-\frac{1}{2} \leq f \leq \frac{1}{2}$.

4) 

$$h_t = \begin{cases} c & \text{for } t = 0, \\ 2c & \text{for } t = 1, \\ 3c & \text{for } t = 2, \\ 4c & \text{for } t = 3, \\ 5c & \text{for } t = 4. \end{cases}$$

Assume $c = 1$ and $T = 4$. Symmetrical threshold (for equal error):

$$k = \left( \sum_{u=0}^{4} h_u s_{T-u} \right) / 2 = \left( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \right) / 2 = 27.5.$$
Error rates: Expected value for received signal $Y_T$:

\[
\begin{align*}
E[Y_T \mid \text{no signal}] &= E[N_4 + 2N_3 + 3N_2 + 4N_1 + 4N_0] = 0 \\
E[Y_T \mid \text{signal}] &= E[\sum_{u=0}^{4} h_u(s_{T-u} + N_{T-u})] = 2k = 55
\end{align*}
\]

Variance for received signal $Y_T$:

\[
\begin{align*}
V[Y_T \mid \text{both for “no signal” and “signal”}] &= V[N_4 + 2N_3 + 3N_2 + 4N_1 + 5N_0] \\
&= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55
\end{align*}
\]

Error probability:

\[
\begin{align*}
P(Y_T > k \mid \text{no signal}) &= P(Y_T < k \mid \text{signal}) \\
&= 1 - P \left( \frac{Y_T - 0}{\sqrt{55}} < \frac{k - 0}{\sqrt{55}} \right) = P \left( \frac{Y_T - 55}{\sqrt{55}} < \frac{k - 55}{\sqrt{55}} \right) \\
&= 1 - \Phi(27.5/\sqrt{55}) = 1 - \Phi(3.708) = 0.044 \cdot 10^{-4}
\end{align*}
\]

5) The probability $P(Y(t) < 5 + 0.5Y'(t)) = P(4X'(t) - 2X''(t) < 5)$. The covariance functions are given by

\[
r_{X'}(\tau) = -r''_X(\tau) = 6e^{-3\tau^2}(1 - 6\tau^2),
\]

and

\[
r_{X''}(\tau) = -r''_X(\tau) = (108 - 1296\tau^2 + 1296\tau^4)e^{-3\tau^2},
\]

and the cross-covariance function between $X'(t)$ and $X''(t)$ is zero. We get $r_{X'}(0) = 6$ and $r_{X''}(0) = 108$ which yields

\[
4X'(t) - 2X''(t)eN(0, \sqrt{16 \cdot 6 + 4 \cdot 108}).
\]

The probability is

\[
P(4X'(t) - 2X''(t) < 5) = \Phi \left( \frac{5}{22.98} \right) \approx 0.586.
\]

6) The cross-covariance is given from

\[
r_{X,Y}(s,t) = C(X_s, Y_t) = C(X_s, 0.4X_t + 0.9X_{t-1} + e_t + 18) \\
&= 0.4C(X_s, X_t) + 0.9C(X_s, X_{t-1}) + 0 \\
&= 0.8 \cdot 0.9^{t-s} + 1.8 \cdot 0.9^{t-1-s} = 0.8 \cdot 0.9^{|\tau|} + 1.8 \cdot 0.9^{|\tau-1|}
\]

with $\tau = t - s$, and $r_X(\tau) = r_Z(\tau) = 2 \cdot 0.9^{|\tau|}$. The Yule-Walker-equations, defined as

\[
r_X(0) + a_1r_X(1) = \sigma^2 \\
r_X(1) + a_1r_X(0) = 0,
\]

gives $a_1 = -0.9$ and $\sigma^2 = 0.38$, and the spectral density for $X_t$ is accordingly

\[
R_X(f) = \frac{\sigma^2}{|1 + a_1e^{-i2\pi f}|^2} = \frac{0.38}{1.81 - 1.8\cos 2\pi f}.
\]

The process $Y_t$ is the output response of the input $X_t$ of the linear filter with the frequency function $H(f) = 0.4 + 0.9e^{-i2\pi f}$, and added white noise. The cross-spectrum is given as

\[
R_{X,Y}(f) = H(f)R_X(f) = 0.38 \cdot \frac{0.4 + 0.9 \cos 2\pi f - i \cdot 0.9 \sin 2\pi f}{1.81 - 1.8 \cos 2\pi f}.
\]