Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-08-31
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are B and F. A spectral density is always positive, which is true for C and E. Thus A and D are realizations.

D varies faster than A, and D then corresponds to the strong spectral peak at $f = 0.25$ in C, where A corresponds to the more low-frequency spectral peak in E. The same difference is seen for the covariances B and F where F is the one with the slowest oscillation. Therefore A, F and E belong together and D, B and C.

The covariance function in B is the MA(6)-process as it is zero for $\tau > 6$. The covariance function in F belongs to the AR(2)-process, as the response certainly has covariance values for $\tau > 6$ and cannot be the MA(6)-process.

2) The spectral density for the input process is

$$R_X(f) = \pi e^{-2\pi|f|},$$

and for the output process

$$R_Y(f) = |H(f)|^2 R_X(f) = \pi e^{-2\pi|f|}, \quad 1 < |f] < 2.$$

The variance is

$$V[Y(t)] = \int_{-\infty}^{\infty} R_Y(f)df = 2\int_1^2 \pi e^{-2\pi|f|}df = e^{-2\pi} - e^{-4\pi}.$$

3) a) The sampling interval d is decided from $\frac{1}{2d} > 1.5$, giving $d < \frac{1}{3}$.

b) Let Z_t denote the sampled process for $d = 1$ with spectral density

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_X(f+k).$$

Aliasing will occur and and for $0 \le f \le \frac{1}{2}$ we have submissions from $k = 0$ and $k = \pm 1$ as

$$\begin{aligned} R_Z(f) &= R_X(f) + R_X(f+1) + R_X(f-1) = 1.5 - f + 1.5 - |f+1| + 1.5 - |f-1| = \\ &= 1.5 - f + 1.5 - f - 1 + 1.5 + f - 1 = 2.5 - f. \end{aligned}$$

This gives $R_Z(f) = 2.5 - |f|$, for $-\frac{1}{2} \le f \le \frac{1}{2}$.

4)

$$h_t = \begin{cases} c & \text{for } t = 0, \\ 2c & \text{for } t = 1, \\ 3c & \text{for } t = 2, \\ 4c & \text{for } t = 3, \\ 5c & \text{for } t = 4. \end{cases}$$

Assume $c = 1$ and $T = 4$. Symmetrical threshold (for equal error):

$$k = \left(\sum_{u=0}^{4} h_u s_{T-u}\right)/2 = \left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right)/2 = 27.5.$$

Error rates: Expected value for received signal Y_T:

$$\begin{aligned} \mathrm{E}[Y_T \mid \text{no signal}] = & \quad \mathrm{E}[N_4 + 2N_3 + 3N_2 + 4N_1 + 4N_0] = 0 \\ \mathrm{E}[Y_T \mid \text{signal}] = & \quad \mathrm{E}[\textstyle\sum_{u=0}^{4} h_u(s_{T-u} + N_{T-u})] = 2k = 55 \end{aligned}$$

Variance for received signal Y_T:

$$\begin{aligned} \mathrm{V}[Y_T \mid \text{both for "no signal" and "signal"}] = & \quad \mathrm{V}[N_4 + 2N_3 + 3N_2 + 4N_1 + 5N_0] \\ = & \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55 \end{aligned}$$

Error probability:

$$\begin{aligned} \mathrm{P}(Y_T > k \mid \text{no signal}) = & \quad \mathrm{P}(Y_T < k \mid \text{signal}) \\ = 1 - \mathrm{P}\left(\frac{Y_T - 0}{\sqrt{55}} < \frac{k-0}{\sqrt{55}}\right) = & \quad \mathrm{P}\left(\tfrac{Y_T - 55}{\sqrt{55}} < \tfrac{k-55}{\sqrt{55}}\right) \\ = 1 - \Phi(27.5/\sqrt{55}) = & \quad 1 - \Phi(3.708) = 1.044 \cdot 10^{-4} \end{aligned}$$

5) The probability $P(Y(t) < 5 + 0.5Y'(t)) = P(4X'(t) - 2X''(t) < 5)$. The covariance functions are given by

$$r_{X'}(\tau) = -r''_X(\tau) = 6e^{-3\tau^2}(1 - 6\tau^2),$$

and

$$r_{X''}(\tau) = -r''_{X'}(\tau) = (108 - 1296\tau^2 + 1296\tau^4)e^{-3\tau^2},$$

and the cross-covariance function between $X'(t)$ and $X''(t)$ is zero. We get $r_{X'}(0) = 6$ and $r_{X''}(0) = 108$ which yields

$$4X'(t) - 2X''(t) \epsilon N(0, \sqrt{16 \cdot 6 + 4 \cdot 108}).$$

The probability is

$$P(4X'(t) - 2X''(t) < 5) = \Phi(\frac{5}{22.98}) \approx 0.586.$$

6) The cross-covariance is given from

$$\begin{aligned} r_{X,Y}(s,t) &= C(X_s, Y_t) = C(X_s, 0.4X_t + 0.9X_{t-1} + e_t + 18) \\ &= 0.4C(X_s, X_t) + 0.9C(X_s, X_{t-1}) + 0 \\ &= 0.8 \cdot 0.9^{|t-s|} + 1.8 \cdot 0.9^{|t-1-s|} = 0.8 \cdot 0.9^{|\tau|} + 1.8 \cdot 0.9^{|\tau-1|} \end{aligned}$$

with $\tau = t - s$, and $r_X(\tau) = r_Z(\tau) = 2 \cdot 0.9^{|\tau|}$. The Yule-Walker-equations, defined as

$$\begin{aligned} r_X(0) + a_1 r_X(1) = \sigma^2 \\ r_X(1) + a_1 r_X(0) = 0, \end{aligned}$$

gives $a_1 = -0.9$ and $\sigma^2 = 0.38$, and the spectral density for X_t is accordingly

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f}|^2} = \frac{0.38}{1.81 - 1.8\cos 2\pi f}.$$

The process Y_t is the output response of the input X_t of the linear filter with the frequency function $H(f) = 0.4 + 0.9e^{-i2\pi f}$, and added white noise. The cross-spectrum is given as

$$R_{X,Y}(f) = H(f)R_X(f) = 0.38 \cdot \frac{0.4 + 0.9\cos 2\pi f - i\,0.9\sin 2\pi f}{1.81 - 1.8\cos 2\pi f}.$$