Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-01-08
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are B and F. A spectral density is always positive, which is true for C and D. Thus A and E are realizations.

E varies faster than A, and E then corresponds to the strong spectral peak at $f = 0.5$ in C, where A corresponds to the more low-frequency spectral peaks in D. The same difference is seen for the covariances B and F where F is the one with the slowest oscillation. Therefore A, F and D belong together and E, B and C.

The covariance function in F belongs to an AR-process, as the response certainly has covariance values for $\tau > 10$. The order is AR(4) as two peaks are seen in the spectral density D. The covariance function in C is an MA(4)-process as it is zero for $\tau > 4$.

2) We have an MA(2)–process

$$r_X(\tau) = \begin{cases} 12, & \tau = 0, \\ -8, & \tau = \pm 1, \\ 2, & \tau = \pm 2, \\ 0, & \text{for other values.} \end{cases}$$

The spectral density is found as

$$R_X(f) = 12 - 16\cos(2\pi f) + 4\cos(4\pi f).$$

3) a) Figure A shows the Hanning window periodogram and B the rectangle window periodogram. The rectangle window has high sidelobes and causes leakage for varying spectra. Then the expected value are not able to follow the true spectrum in low power frequency regions. The Hanning window has a broader mainlobe (twice as wide as the rectangle window)and is not able to resolve closely spaced peaks.

b) The variance will be approximately K times smaller.

4) a) In the triangle, every corner has two neighbors at the distance L_T. In the square every corner has two neighbors at the distance L_K and one neighbor at the distance $\sqrt{2}\,L_T$. The variances are

$$\begin{aligned} V(X_T) &= \frac{1}{9}V(X_1 + X_2 + X_3) = \frac{1}{9}(3r(0) + 3 \cdot 2\,r(L_T)) = (1 + 2e^{-L_T})/3 \\ V(Y_K) &= \frac{1}{16}V(Y_1 + Y_2 + Y_3 + Y_4) = \frac{1}{16}(4r(0) + 4 \cdot 2\,r(L_K) + 4\,r(\sqrt{2}L_K)) \\ &= (1 + 2e^{-L_K} + e^{-\sqrt{2}L_K})/4. \end{aligned}$$

b) If the walk should be at most 1 km for one lap, the triangle side is $L_T = 1/3$ and the square side $L_K = 1/4$. The variances are

$$V(X_T) = (1 + 2e^{-1/3})/3 = 0.81102, \quad V(Y_K) = (1 + 2e^{-1/4} + e^{-\sqrt{2}/4})/4 = 0.81495,$$

i.e., the triangle gives the smallest variance.

c) If the walk should be at most 4 km, $L_T = 4/3$ and $L_K = 1$ giving the variances

$$V(X_T) = (1 + 2e^{-4/3})/3 = 0.50906, \qquad V(Y_K) = (1 + 2e^{-1} + e^{-\sqrt{2}})/4 = 0.49472,$$

i.e., the square gives the smallest.

5) a) The spectral density of $X(t)$ is $R_X(f) = \frac{6}{1+(2\pi f)^2}$. The impulse response of the filter is

$$h(t) = \delta(t) - \frac{1}{2}e^{-t/2}, \qquad t \geq 0,$$

and the frequency function

$$H(f) = 1 - \frac{1}{2}\int_0^\infty e^{-i2\pi f u}\, e^{-u/2}\, du = 1 - \frac{1}{1 + i4\pi f} = \frac{i4\pi f}{1 + i4\pi f}.$$

The spectral density of $Y(t)$ is $R_Y(f) = |H(f)|^2\, R_X(f)$, i.e.

$$R_Y(f) = \frac{6}{1 + (2\pi f)^2} \cdot \frac{(4\pi f)^2}{1 + (4\pi f)^2}.$$

b) From

$$R_Y(f) = \frac{6}{1 + (2\pi f)^2} \cdot \frac{(4\pi f)^2}{1 + (4\pi f)^2} = \frac{8}{1 + (2\pi f)^2} - \frac{8}{1 + (4\pi f)^2},$$

the covariance function becomes

$$r_Y(t) = 4e^{-|t|} - 2e^{-|t|/2}.$$

c) As $Y(5)$ is Gaussian distributed with expected value $E[Y(t)] = 0$ and $V[Y(t)] = r_Y(0) = 2$, $P(Y(5) > 2) = 1 - \Phi((2-0)/\sqrt{2}) = 1 - \Phi(1.4142) = 1 - 0.9213 = 0.0787$.

6) a) The expected value is given as

$$E[I] = a(\int_0^1 E[X(t)]dt + \int_2^3 E[Y(t)]dt) + b\int_1^2 E[X(t)+Y(t)]dt = a\cdot 2m + b\cdot 2m = m,$$

which results in $a + b = 1/2$.

b) The variance is given as

$$V[\int_0^1 X(t)dt] = 2\int_0^1 \int_0^t 2e^{-(t-s)}ds\; dt = 4\int_0^1 [e^{-t-s}]_0^t dt$$

$$= 4\int_0^1 1 - e^{-t}dt = 4[t + e^{-t}]_0^1 = 4(1 + e^{-1} - 1) = 4e^{-1}.$$

c) The covariance is given from

$$C\left[\int_0^1 X(t)dt, \int_1^2 X(t)dt\right] = \int_0^1 \int_1^2 r(t-s)dt\; ds = \int_1^2 \int_0^1 2e^{-(t-s)}ds\; dt =$$

$$2\int_1^2 [e^{-(t+s)}]_0^1 dt = 2\int_1^2 e^{-(t-1)} - e^{-t}dt = 2((e^{-2} - e^{-1}) - (e^{-1} - 1)) = 2(e^{-1} - 1)^2.$$

d) The variance

$$V[I] = a^2 V[\int_0^1 X(t)dt] + b^2 V[\int_1^2 X(t)dt] + 2ab \int_0^1 \int_1^2 r(t-s)dt\, ds +$$

$$a^2 V[\int_2^3 Y(t)dt] + b^2 V[\int_1^2 Y(t)dt] + 2ab \int_2^3 \int_1^2 r(t-2)dt\, ds =$$

$$(2a^2 + 2b^2)4e^{-1} + 4ab \cdot 2(1-e^{-1})^2.$$

With $a + b = 0.5$ we get

$$V(I) = (4a^2 - 2a + 0.5)4e^{-1} + 4(a - 2a^2)(1-e^{-1})^2.$$

Differentiation with respect to a yields

$$\frac{\partial V(I)}{\partial a} = a(32e^{-1} - 16(1-e^{-1})^2 - 8e^{-1} + 4(1-e^{-1})^2 = 0,$$

resulting in

$$\begin{aligned} a &= \frac{1}{4} \\ b &= \frac{1}{4}. \end{aligned}$$