Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2018-01-08
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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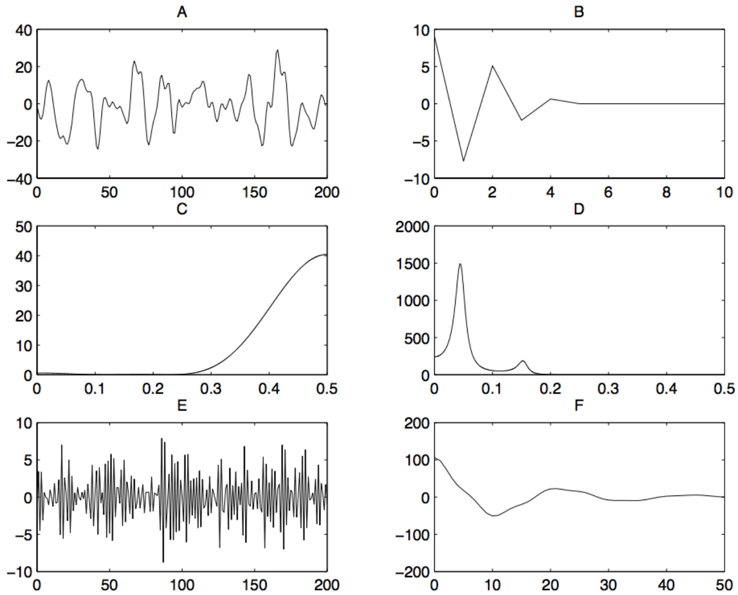
Solutions, correct and well motivated, of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits. The maximal number of credits are 90. The limit for passing the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) The following figures show realizations, covariance functions and spectral densities of one AR(p)-process and one MA(q)-process. Determine, with a motivation, what figures that are realizations, covariance functions and spectral densities. Also state which realization, covariance function and spectral density that are connected. Decide of which type the different processes are (AR or MA) and which orders they have. The order can be assumed to be less than 10 for both processes. Important! To receive full number of credits correct motivations have to be given for all answers.

(10p)



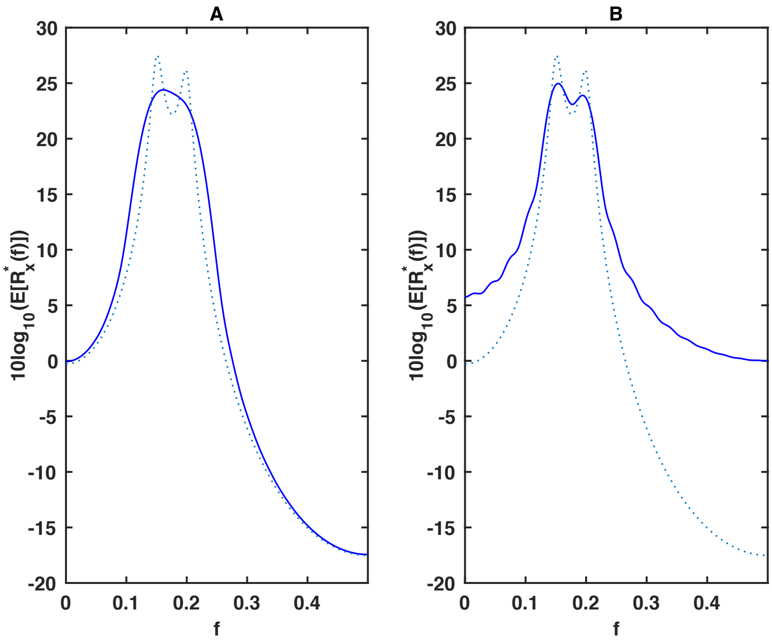
2) Låt e_t, $t = 0, \pm 1, \pm 2, \ldots$ be a sequence of independent stochastic variables with $E[e_t] = 0$ and $V[e_t] = 2$. Let

$$X_t = e_t - 2e_{t-1} + e_{t-2}.$$

Determine the covariance function and the spectral density for X_t.

(10p)

3) Two spectral estimates of the sequence, $x(0), \ldots, x(n-1)$, which are observations from the stochastic process $X(t)$ with spectral density $R_x(f)$, are calculated.



a) In the figures above, the expected values of the periodogram using two different window functions, the rectangle window and the Hanning window, are illustrated with solid lines. The dotted line shows the true spectral density. The estimates are based on the same data sequence length n in both cases. Which of the figures shows the expected value when the rectangle window is used and which show the expected value when the Hanning window is used. Motivate your answer.

b) The sequence is divided into K sequences, a periodogram is computed for each shorter sequence and the final estimate is calculated as the average of the K periodograms. How much does the variance decrease in comparison to the periodogram in a)?

(10p)

4) We would like to estimate the average level of a pollution of a surface with use of the average value at a number of measurement points. The following alternatives are possible:

Triangle: Use three measurements at the corners of a triangle with side lengths L_T.

Square: Use four measurements at the corners of a square with side lengths L_K.

We suppose that the covariance function is e^{-d}, where d is the distance in kilometers.

a) Write down the expression for the variance of the estimate of the average level of the two cases as functions of L_T and L_K, respectively.

(13p)

For exercise 4b) and c), see next page.

b) When the measurements are collected, we walk along the sides of the triangle and the square, respectively. Which of the methods give the smallest variance if we just are able to walk 1 kilometer, i.e., if the total distance around the triangle is 1 km and around the square 1 km?

(5p)

c) If we instead are able to walk 4 kilometers, i.e., if the distance around the triangle is 4 km and around the square 4 km, which of the methods give the smallest variance?

(2p)

5) A stationary Gaussian process $X(t), t \in \mathbb{R}$ with expected value $E[X(t)] = 0$ has the covariance function $r_X(\tau) = 3\,e^{-|\tau|}$. The process $X(t)$ is the input of a linear time-invariant filter giving the output process $Y(t)$ according to

$$Y(t) = X(t) - \frac{1}{2}\int_0^\infty e^{-u/2}\,X(t-u)\,du.$$

a) Calculate the spectral density $R_Y(f)$.

(10p)

b) Compute the covariance function $r_Y(\tau)$.

(5p)

c) Determine the probability $P(Y(5) > 2)$.

(5p)

6) Two temperature sensors measure the temperature during different time intervals, $X(t)$ during the interval $0 \le t \le 2$ and $Y(t)$ during the interval $1 \le t \le 3$. Assume the two processes to be stationary and independent with the same expected value m and the covariance function

$$r_X(\tau) = r_Y(\tau) = 2\,e^{-|\tau|}.$$

One wants to use the measurements to get the best possible estimate of the temperature and therefore the measurements are combined as

$$I = a \cdot \{\int_0^1 X(t)dt + \int_2^3 Y(t)dt\} + b \cdot \int_1^2 (X(t) + Y(t))dt.$$

a) Determine a condition for the constants a and b so that the estimator I get the expected value m.

(3p)

b) Calculate $V\left[\int_0^1 X(t)dt\right]$.

(4p)

c) Calculate $C\left[\int_0^1 X(t)dt, \int_1^2 X(t)dt\right]$.

(5p)

d) Calculate a and b so that the estimator I becomes unbiased and gets as small variance as possible. (Hint: Use your calculations in a), b) and c).)

(8p)