1) a) The pole-zero plot in A has a zero at the unit circle at angle $\omega = 2\pi 0.3$ and a pole close to the unit circle at $\omega = 2\pi 0.2$. The corresponding spectral estimate should then have a zero at $f = 0.3$ and a peak at $f = 0.2$, which is figure 3. The pole-zero plot in B has only two poles and is most likely to belong to figure 2. The pole-zero plot in C, that has a pole at $\omega = 2\pi 0.3$ and a zero at the unit circle at angle $\omega = 2\pi 0.2$, belong to the spectral estimate in figure 1. Process A-3 is an ARMA(2,2), B-2 is an AR(2) and C-1 an ARMA(2,2).

b) The periodogram variance is approximately $R_X^2(f)$ and the reduction of the Welch method is assumed to be the number of possible windows. The variance is approximately $V[R_x(f)] = \frac{R_x^2(f)}{C}$ as $0.5 \cdot 256 \cdot 7 + 128 = 1024$.

2) The order of the MA-process is $q = 3$ and the spectral density is

$$R_X(f) = \sum_\tau r_X(\tau) e^{-i2\pi f \tau} = 2 + 2 \cos(6\pi f), \quad |f| \leq 0.5.$$ 

The spectral density is also given from

$$R_X(f) = |\sum_{k=0}^q b_k e^{-i2\pi f k}|^2,$$

which gives $b_0 = b_3 = 1$ and all other $b_k = 0$.

3) a) Correct. Transform table with $\alpha = 1$ gives $r_X(\tau) = 1/(1 + \tau^2)$.

b) Correct. The process is differentiable as the covariance function $r_X(\tau)$ is twice differentiable, giving $r_X'(\tau) = -2\tau/(1 + \tau^2)^2$ and $r_X''(\tau) = r_X'/(1 + \tau^2)^2 = \frac{8\tau^2}{(1 + \tau^2)^4} - \frac{2}{(1 + \tau^2)^3}$.

c) Wrong. The variance $V[(X(0) + X(t))/2] = \frac{1}{4} (V[X(0)] + V[X(t)] + 2C[X(0), X(t)]) = \frac{1}{4} (1 + 1 + \frac{2}{1 + \frac{1}{2}}) \rightarrow \frac{1}{2}$ dà $t \rightarrow \infty$.

d) Wrong. The output spectral density is $R_y(f) = |1 + if|^2 \pi e^{-2\pi|f|} = (1 + f^2) \pi e^{-2\pi|f|}$.

e) Correct. The derivative of a Gaussian process, if it exists, is a Gaussian process and a linear combination of Gaussian processes is a Gaussian process.

4) a) The Yule-Walker-equations give

$$
\begin{align*}
12 + 8a_1 + 2a_2 &= \sigma^2 \\
8 + 12a_1 + 8a_2 &= 0 \\
2 + 8a_1 + 12a_2 &= 0,
\end{align*}
$$

with the solution $a_1 = -1, a_2 = 0.5$ and $\sigma^2 = 5$. The spectral density function is

$$R_X(f) = \frac{5}{|1 - e^{-i2\pi f} + 0.5e^{-4\pi f}|^2} = \frac{5}{2.25 - 3\cos(2\pi f) + \cos(4\pi f)}.$$
b) The spectral density of the noise disturbance is $R_N(f) = 4$. The Wiener filter becomes

$$H(f) = \frac{R_X(f)}{R_X(f) + R_N(f)} = \frac{5}{5 + 4(2.25 - 3\cos(2\pi f) + \cos(4\pi f))} = \frac{14 - 12\cos(2\pi f) + 4\cos(4\pi f)}{5}.$$  

5) a) The covariance function is,

$$r_Y(\tau) = r_X(\tau) = \int_{-2}^{2} e^{i2\pi f\tau} df + \int_{0}^{2} e^{i2\pi f\tau} df =, $$

$$= \frac{\sin(2\pi 2.2\tau)}{\pi \tau} - \frac{\sin(2\pi 2.0\tau)}{\pi \tau}, \quad \tau \neq 0$$

and $r_Y(0) = 0.4$.

b) The resulting spectral densities are

$$R_Z(f) = \begin{cases} 0 & \text{for } |f| \leq 1.8, \\ 1 & \text{for } 1.8 < |f| \leq 2.0. \end{cases}$$

when $d = 1/4$ and

$$R_Z(f) = \begin{cases} 1 & \text{for } |f| \leq 0.2, \\ 0 & \text{for } |f| > 0.2. \end{cases}$$

when $d = 1/2$.

c) For large $n$

$$V[\hat{m}_n] \to \frac{1}{n} \sum_{\tau=-n}^{n} r_Z(\tau),$$

where $r_Z(\tau)$ is the sampled $r_Y(\tau)$ at $\tau = 0, \pm d, \pm 2d, \ldots$. From the definition of spectral density we identify $R_Z(0) = \sum_{\tau=-n}^{n} r_Z(\tau)e^{-i2\pi n\tau} = \sum_{\tau=-n}^{n} r_Z(\tau)$. Then we get

$$nV[\hat{m}_n] = (1/d)R_Z(0),$$

where $1/d$ relates the value of $R_Z(0)$ to the spectral density for normalized frequency and thereby to the correct summation of the discrete-time covariance function. The smallest $R_Z(0)$ is found to be zero when $d = 1/4$.

6) a) As $r_Y(\tau) = r_X(\tau + \sigma^2)$ for $\tau \leq h/2$ and $r_Y(\tau) = r_X(\tau)$ for $\tau > h/2$, the variance $V[Z(t)]$ is

$$V\left[\frac{aY(t) - bY(t - h)}{h}\right] = \frac{(a^2 + b^2)r_Y(0) - 2abr_Y(h)}{h^2} = \frac{(a^2 + b^2)(1 + \sigma^2) - 2ab e^{-h^2/2}}{h^2}.$$  

b) The cross-covariance, $C[X'(t), Z(t + \tau)]$ is

$$C[X'(t), \frac{aY(t + \tau) - bY(t + \tau - h)}{h}] = \frac{a}{h} C[X'(t), Y(t+\tau)] - \frac{b}{h} C[X'(t), Y(t+\tau-h)] =$$

$$-\frac{a}{h} r_X(\tau) + \frac{b}{h} r_X(\tau - h) = \frac{a}{h} \tau e^{-\tau^2/2} - \frac{b}{h} (\tau - h) e^{-(\tau-k)^2/2},$$

with $C[X'(t), Y(t+\tau)] = C[X'(t), X(t+\tau)] = -r_X'(\tau)$. 

2
c) We find
\[ E[(X'(t) - Z(t))^2] = V[X'(t) - Z(t)] = V[X'(t)] + V[Z(t)] - 2C[X'(t), Z(t)]. \]

With
\[ V[X'(t)] = r_X'(0) = -r''_X(0) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2} = 1, \]
and the expressions in a) and b) we get
\[ E[(X'(t) - Z(t))^2] = 1 + \left( a^2 + b^2 \right) \frac{1 + \sigma^2}{h^2} - 2abe^{-h^2/2}/h^2 - 2be^{-h^2/2}. \]

For \( \sigma^2 = 1 \) and \( h = 0.5 \)
\[ E[(X'(t) - Z(t))^2] = 1 + 8(a^2 + b^2) - \frac{8abe^{-1/8}}{h} - \frac{2be^{-1/8}}{h}. \]

The minimum is found through derivation with respect to \( a \) and \( b \)
\[ 16a - 8be^{-1/8} = 0, \]
\[ 16b - 8ae^{-1/8} - 2e^{-1/8} = 0, \]

with resulting
\[ a = \frac{e^{-1/4}}{16 - 4e^{-1/4}} \approx 0.0604 \quad b = \frac{e^{-1/8}}{8 - 2e^{-1/4}} \approx 0.1370. \]