1) a) The pole-zero plot in A has a zero at the unit circle at angle \( \omega = 2\pi 0.3 \) and a pole close to the unit circle at \( \omega = 2\pi 0.2 \). The corresponding spectral estimate should then have a zero at \( f = 0.3 \) and a peak at \( f = 0.2 \), which is figure 3. The pole-zero plot in B has only two poles and is most likely to belong to figure 2. The pole-zero plot in C, that has a pole at \( \omega = 2\pi 0.3 \) and a zero at the unit circle at angle \( \omega = 2\pi 0.2 \), belong to the spectral estimate in figure 1. Process A-3 is an ARMA(2,2), B-2 is an AR(2) and C-1 an ARMA(2,2).

b) The periodogram variance is approximately \( R_X(f) \) and the reduction of the Welch method is assumed to be the number of possible windows. The variance is approximately \( V[\hat{R}_X(f)] = \frac{R_X(f)}{f} \) as \( 0.5 \cdot 256 \cdot 7 + 128 = 1024 \).

2) The order of the MA-process is \( q = 3 \) and the spectral density is

\[
R_X(f) = \sum_{\tau} r_X(\tau) e^{-i2\pi f \tau} = 2 + 2 \cos(6\pi f), \quad |f| \leq 0.5.
\]

The spectral density is also given from

\[
R_X(f) = \left| \sum_{k=0}^{q} b_k e^{-i2\pi f k} \right|^2,
\]

which gives \( b_0 = b_3 = 1 \) and all other \( b_k = 0 \).

3) a) Correct. Transform table with \( \alpha = 1 \) gives \( r_X(\tau) = 1/(1 + \tau^2) \).

b) Correct. The process is differentiable as the covariance function \( r_X(\tau) \) is twice differentiable, giving \( r_X'(\tau) = -2\tau/(1 + \tau^2)^2 \) and \( r_X''(\tau) = \frac{8\tau^2}{(1 + \tau^2)^3} - \frac{2}{(1 + \tau^2)^2} \).

c) Wrong. The variance \( V[(X(0) + X(t))/2] = \frac{1}{4}(V[X(0)] + V[X(t)] + 2C[X(0), X(t)]) = \frac{1}{4}(1 + 1 + \frac{2}{1 + \tau^2}) \rightarrow \frac{1}{2} \text{ as } t \rightarrow \infty.

d) Wrong. The output spectral density is \( R_y(f) = |1 + if|^2pe^{-2\pi|f|} = (1 + f^2)p e^{-2\pi|f|} \).

e) Correct. The derivative of a Gaussian process, if it exists, is a Gaussian process and a linear combination of Gaussian processes is a Gaussian process.

4) a) The Yule-Walker-equations give

\[
\begin{align*}
12 + 8a_1 + 2a_2 &= \sigma^2 \\
8 + 12a_1 + 8a_2 &= 0 \\
2 + 8a_1 + 12a_2 &= 0,
\end{align*}
\]

with the solution \( a_1 = -1, \ a_2 = 0.5 \) and \( \sigma^2 = 5 \). The spectral density function is

\[
R_X(f) = \frac{5}{|1 - e^{-i2\pi f} + 0.5e^{-i4\pi f}|^2} = \frac{5}{2.25 - 3 \cos(2\pi f) + \cos(4\pi f)}.
\]
b) The spectral density of the noise disturbance is $R_N(f) = 4$. The Wiener filter becomes

\[
H(f) = \frac{R_X(f)}{R_X(f) + R_N(f)} = \frac{5}{5 + 4(2.25 - 3\cos(2\pi f) + \cos(4\pi f))} = \frac{14 - 12\cos(2\pi f) + 4\cos(4\pi f)}{5}.
\]

5) a) The covariance function is,

\[
r_Y(\tau) = r_X(\tau) = \int_{-2.2}^{2.2} e^{i2\pi f \tau} df + \int_{2.2}^{2.0} e^{i2\pi f \tau} df =, \]

\[
= \frac{\sin(2\pi 2.2\tau)}{\pi \tau} - \frac{\sin(2\pi 2.0\tau)}{\pi \tau}, \quad \tau \neq 0
\]

and $r_Y(0) = 0.4$.

b) The resulting spectral densities are

\[
R_Z(f) = \begin{cases} 0 & \text{for } |f| \leq 1.8, \\ 1 & \text{for } 1.8 < |f| \leq 2.0. \end{cases}
\]

when $d = 1/4$ and

\[
R_Z(f) = \begin{cases} 1 & \text{for } |f| \leq 0.2, \\ 0 & \text{for } |f| > 0.2. \end{cases}
\]

when $d = 1/2$.

c) For large $n$

\[
V[\hat{m}_n] \to \frac{1}{n} \sum_{\tau=-n}^{n} r_Y(\tau).
\]

From the definition of spectral density we identify $R_Y(0) = \sum_{\tau=-n}^{n} r_Y(\tau)e^{-i2\pi0\tau} = \sum_{\tau=-n}^{n} r_Y(\tau)$. Then we get $nV[\hat{m}_n] = R_Y(0) = (1/d)R_Z(0)$, where $R_Z(0) = 0$ when $d = 1/4$ and $R_Z(0) = 1$ when $d = 1/2$. The sampling distance $d = 1/4$ thereby gives the smallest variance which is preferable.

6) a) As $r_Y(\tau) = r_X(\tau) + \sigma^2$ for $\tau \leq h/2$ and $r_Y(\tau) = r_X(\tau)$ for $\tau > h/2$, the variance $V[Z(t)]$ is

\[
V[\frac{aY(t) - bY(t - h)}{h}] = \frac{a^2 + b^2}{h^2} = \frac{(a^2 + b^2)(1 + \sigma^2) - 2abe^{-h^2/2}}{h^2}.
\]

b) The cross-covariance, $C[X'(t), Z(t + \tau)]$ is

\[
C[X'(t), \frac{aY(t + \tau) - bY(t + \tau - h)}{h}] = \frac{a}{h} C[X'(t), Y(t+\tau)] - \frac{b}{h} C[X'(t), Y(t+\tau-h)] =
\]

\[
-\frac{a}{h} r_X'(\tau) + \frac{b}{h} r_X'(\tau - h) = \frac{a}{h} \tau e^{-\tau^2/2} - \frac{b}{h} (\tau - h) e^{-(\tau-h)^2/2},
\]

with $C[X'(t), Y(t + \tau)] = C[X'(t), X(t + \tau)] = -r_X'(\tau)$. 

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c) We find
\[ E[(X'(t) - Z(t))^2] = V[X'(t) - Z(t)] = V[X'(t)] + V[Z(t)] - 2C[X'(t), Z(t)]. \]

With
\[ V[X'(t)] = r_{X'}(0) = -r''_{X}(0) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2} = 1, \]
and the expressions in a) and b) we get
\[ E[(X'(t) - Z(t))^2] = 1 + (a^2 + b^2)(1 + \sigma^2)/h^2 - 2abe^{-h^2/2}/h^2 - 2be^{-h^2/2}. \]

For \( \sigma^2 = 1 \) and \( h = 0.5 \)
\[ E[(X'(t) - Z(t))^2] = 1 + 8(a^2 + b^2) - 8abe^{-1/8}/h^2 - 2be^{-1/8}. \]

The minimum is found through derivation with respect to \( a \) and \( b \)
\[
\begin{align*}
16a - 8be^{-1/8} &= 0, \\
16b - 8ae^{-1/8} - 2e^{-1/8} &= 0,
\end{align*}
\]
with resulting
\[
\begin{align*}
a &= \frac{e^{-1/4}}{16 - 4e^{-1/4}} \approx 0.0604, \\
b &= \frac{e^{-1/8}}{8 - 2e^{-1/4}} \approx 0.1370.
\end{align*}
\]