Mathematical Statistics, Centre for Mathematical Sciences, Lund University
Exam in Stationary stochastic processes (FMSF10/MASC04)
Date: 2017-10-28
Time: 8.00 – 13.00
Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.

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1) a) The pole-zero plot in **A** has a zero at the unit circle at angle $\omega = 2\pi 0.3$ and a pole close to the unit circle at $\omega = 2\pi 0.2$. The corresponding spectral estimate should then have a zero at $f = 0.3$ and a peak at $f = 0.2$, which is figure **3**. The pole-zero plot in **B** has only two poles and is most likely to belong to figure **2**. The pole-zero plot in **C**, that has a pole at $\omega = 2\pi 0.3$ and a zero at the unit circle at angle $\omega = 2\pi 0.2$, belong to the spectral estimate in figure **1**. Process **A-3** is an ARMA(2,2), **B-2** is an AR(2) and **C-1** an ARMA(2,2).

b) The periodogram variance is approximately $R_X^2(f)$ and the reduction of the Welch method is assumed to be the number of possible windows. The variance is approximately $V[\hat{R}_x(f)] = \frac{R_X^2(f)}{7}$ as $0.5 \cdot 256 \cdot 7 + 128 = 1024$.

2) The order of the MA-process is $q = 3$ and the spectral density is

$$R_X(f) = \sum_\tau r_X(\tau) e^{-i2\pi f\tau} = 2 + 2\cos(6\pi f), \quad |f| \leq 0.5.$$

The spectral density is also given from

$$R_X(f) = |\sum_{k=0}^{q} b_k e^{-i2\pi f k}|^2,$$

which gives $b_0 = b_3 = 1$ and all other $b_k = 0$.

3) a) Correct. Transform table with $\alpha = 1$ gives $r_X(\tau) = 1/(1+\tau^2)$.

b) Correct. The process is differentiable as the covariance function $r_X(\tau)$ is twice differentiable, giving $r'_X(\tau) = -2\tau/(1+\tau^2)^2$ and $r_{X'}(\tau) = r''_X(\tau) = \frac{8\tau^2}{(1+\tau^2)^3} - \frac{2}{(1+\tau^2)^2}$.

c) Wrong. The variance $V[(X(0)+X(t))/2] = \frac{1}{4}(V[X(0)]+V[X(t)]+2C[X(0),X(t)]) = \frac{1}{4}(1+1+\frac{2}{1+\tau^2}) \to \frac{1}{2}$ då $t \to \infty$.

d) Wrong. The output spectral density is $R_y(f) = |1+if|^2 \pi e^{-2\pi|f|} = (1+f^2)\pi e^{-2\pi|f|}$.

e) Correct. The derivative of a Gaussian process, if it exists, is a Gaussian process and a linear combination of Gaussian processes is a Gaussian process.

4) a) The Yule-Walker-equations give

$$\begin{aligned} 12 + 8a_1 + 2a_2 &= \sigma^2 \\ 8 + 12a_1 + 8a_2 &= 0 \\ 2 + 8a_1 + 12a_2 &= 0, \end{aligned}$$

with the solution $a_1 = -1$, $a_2 = 0.5$ and $\sigma^2 = 5$. The spectral density function is

$$R_X(f) = \frac{5}{|1 - e^{-i2\pi f} + 0.5e^{-i4\pi f}|^2} =$$

$$\frac{5}{2.25 - 3\cos(2\pi f) + \cos(4\pi f)}.$$

b) The spectral density of the noise disturbance is $R_N(f) = 4$. The Wiener filter becomes

$$\begin{aligned} H(f) &= \frac{R_X(f)}{R_X(f) + R_N(f)} \\ &= \frac{5}{5 + 4(2.25 - 3\cos(2\pi f) + \cos(4\pi f))} \\ &= \frac{5}{14 - 12\cos(2\pi f) + 4\cos(4\pi f)}. \end{aligned}$$

5) a) The covariance function is,

$$r_Y(\tau) = r_X(\tau) = \int_{-2.2}^{-2.0} e^{i2\pi f\tau} df + \int_{2.0}^{2.2} e^{i2\pi f\tau} df =,$$

$$= \frac{\sin(2\pi 2.2\tau)}{\pi\tau} - \frac{\sin(2\pi 2.0\tau)}{\pi\tau}, \quad \tau \neq 0$$

and $r_Y(0) = 0.4$.

b) The resulting spectral densities are

$$R_Z(f) = \begin{cases} 0 & \text{for} \quad |f| \le 1.8, \\ 1 & \text{for} \quad 1.8 < |f| \le 2.0. \end{cases}$$

when $d = 1/4$ and

$$R_Z(f) = \begin{cases} 1 & \text{for} \quad |f| \le 0.2, \\ 0 & \text{for} \quad |f| > 0.2. \end{cases}$$

when $d = 1/2$.

c) For large n

$$V[\widehat{m}_n] \to \frac{1}{n} \sum_{\tau=-n}^{n} r_Z(\tau),$$

where $r_Z(\tau)$ is the sampled $r_Y(\tau)$ at $\tau = 0, \pm d, \pm 2d, \ldots$. From the definition of spectral density we identify $R_Z(0) = \sum_{\tau=-n}^{n} r_Z(\tau)e^{-i2\pi 0\tau} = \sum_{\tau=-n}^{n} r_Z(\tau)$. Then we get

$$nV[\widehat{m}_n] = (1/d)R_Z(0),$$

where $1/d$ relates the value of $R_Z(0)$ to the spectral density for normalized frequency and thereby to the correct summation of the discrete-time covariance function. The smallest $R_Z(0)$ is found to be zero when $d = 1/4$.

6) a) As $r_Y(\tau) = r_X(\tau) + \sigma^2$ for $\tau \le h/2$ and $r_Y(\tau) = r_X(\tau)$ for $\tau > h/2$, the variance $V[Z(t)]$ is

$$V[\frac{aY(t) - bY(t-h)}{h}] = \frac{(a^2 + b^2)r_Y(0) - 2ab r_Y(h)}{h^2} = \frac{(a^2 + b^2)(1 + \sigma^2) - 2abe^{-h^2/2}}{h^2}.$$

b) The cross-covariance, $C[X'(t), Z(t+\tau)]$ is

$$C[X'(t), \frac{aY(t+\tau) - bY(t+\tau-h)}{h}] = \frac{a}{h}C[X'(t), Y(t+\tau)] - \frac{b}{h}C[X'(t), Y(t+\tau-h)] =$$

$$-\frac{a}{h}r'_X(\tau) + \frac{b}{h}r'_X(\tau - h) = \frac{a}{h}\tau e^{-\tau^2/2} - \frac{b}{h}(\tau - h)e^{-(\tau-h)^2/2},$$

with $C[X'(t), Y(t+\tau)] = C[X'(t), X(t+\tau)] = -r'_X(\tau)$.

c) We find

$$E[(X'(t) - Z(t))^2] = V[X'(t) - Z(t)] = V[X'(t)] + V[Z(t)] - 2C[X'(t), Z(t)].$$

With

$$V[X'(t)] = r_{X'}(0) = -r''_X(0) = e^{-\tau^2/2} - \tau^2 e^{-\tau^2/2} = 1,$$

and the expressions in a) and b) we get

$$E[(X'(t) - Z(t))^2] = 1 + (a^2 + b^2)(1 + \sigma^2)/h^2 - 2abe^{-h^2/2}/h^2 - 2be^{-h^2/2}.$$

For $\sigma^2 = 1$ and $h = 0.5$

$$E[(X'(t) - Z(t))^2] = 1 + 8(a^2 + b^2) - 8abe^{-1/8} - 2be^{-1/8}.$$

The minimum is found through derivation with respect to a and b

$$16a - 8be^{-1/8} = 0,$$
$$16b - 8ae^{-1/8} - 2e^{-1/8} = 0,$$

with resulting

$$a = \frac{e^{-1/4}}{16 - 4e^{-1/4}} \approx 0.0604 \quad b = \frac{e^{-1/8}}{8 - 2e^{-1/4}} \approx 0.1370.$$