**Mathematical Statistics, Centre for Mathematical Sciences, Lund University**
**Exam in Stationary stochastic processes (FMSF10/MASC04)**
**Date: 2017-10-28**
**Time: 8.00 – 13.00**
**Help: Calculator, table of formulas in stationary stochastic processes, small table of formulas in mathematics.**

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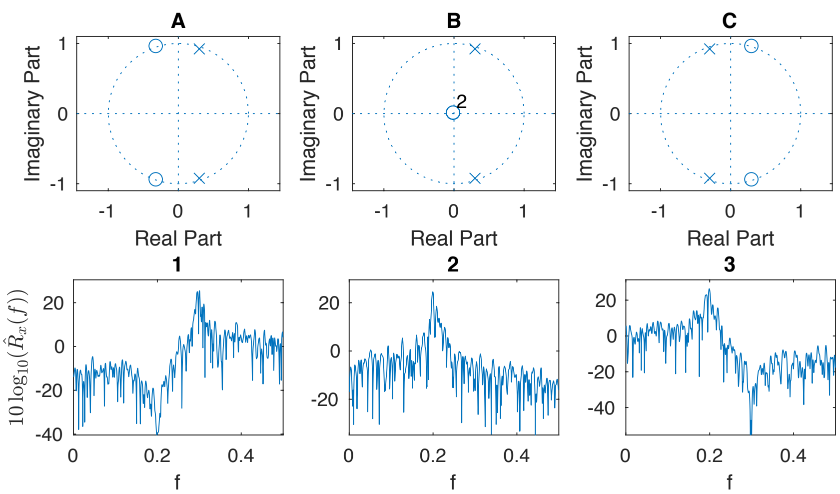
Solutions, correct and well motivated, of exercise 1-3 give 10 credits, and of exercise 4-6 give 20 credits. The maximal number of credits are 90. The limit for passing the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

**1)** a) The figure shows the pole-zero plots of three different discrete time models of stationary processes and modified periodogram estimates of one realization from each model. A Hanning window is used as data window and the spectral estimates are presented in dB-scale. Combine the pole-zero plot with the corresponding spectral estimate. Motivate your answer. Also state which type of model (AR, MA or ARMA) in each case and the corresponding order.

(6p)




b) We have access to 1024 samples of data, where the true spectral density function is $R_X(f)$. The Welch method, with window length 256 in each periodogram and 50% overlap between windows, is applied. Approximately how large is the variance for $0 < |f| < 0.5$? Motivate your answer.

(4p)

**2)** An MA(q)-process $X_t$, $t = 0, \pm1, \pm2, \ldots$, is defined as

$$X_t = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \ldots + b_q e_{t-q},$$

where $e_t$ is white Gaussian noise with $V[e_t] = 1$. The covariance function is $r_X(0) = 2$, $r_X(\pm 3) = 1$ and zero for all other values of $\tau$. Determine the order of the process $q$, the parameters $b_1 \ldots b_q$ and the spectral density.

(10p)

**3)** A zero-mean, weakly stationary, Gaussian process $X(t)$, $t \in \mathbb{R}$, has the spectral density

$$R_X(f) = \pi e^{-2\pi|f|}.$$

State, with a short motivation, for each of the following statements if it is right or wrong. Each correct answer with correct motivation gives 2 credits.

a) The covariance function is

$$r_X(\tau) = \frac{1}{1+\tau^2}.$$

b) The process is differentiable in quadratic mean.

c) The variance

$$V[\frac{(X(0)+X(t))}{2}]$$

approaches zero when $t \to \infty$.

d) After filtering through a linear filter with frequency function

$$H(f) = 1 + if$$

the spectral density of the process is constant for all frequencies.

e) A new process

$$Y(t) = 3X(t) - 2X'(t)$$

is created. The process $Y(t)$ is a Gaussian process.

(10p)

**4)** For an AR(2)-process $X_t$, $t = 0, \pm 1, \pm 2, \ldots$, the following covariance values are given,

$$r_X(\tau) = \begin{cases} 12 & \tau = 0, \\ 8 & |\tau| = 1, \\ 2 & |\tau| = 2, \\ -2 & |\tau| = 3. \end{cases}$$

a) Determine the spectral density of the AR(2)-process. For full number of credits, a real-valued expression should be presented.

(14p)

b) Compute the frequency function of a Wiener filter which reconstructs the AR(2)-process $X_t$ from the measurement data $Y_t = X_t + N_t$, where $N_t$, $t = 0, \pm 1, \pm 2, \ldots$, is a Gaussian white noise sequence with $E[N_t] = 0$ and $V[N_t] = 4$. Note that the expression should be simplified as far as possible and written in real-valued form.

(6p)

**5)** We define the weakly stationary Gaussian process $Y(t) = m + X(t)$, $t \in \mathbb{R}$, where $m$ is an unknown constant expected value. The process $X(t)$ is assumed to have $E[X(t)] = 0$ and spectral density

$$R_X(f) = \begin{cases} 1 & \text{for} \quad |f - 2.1| \leq 0.1, \\ 1 & \text{for} \quad |f + 2.1| \leq 0.1 \\ 0 & \text{otherwise.} \end{cases}$$

An estimate of $m$ should be found as

$$\widehat{m}_n = \frac{Y(d) + Y(2d) + \dots Y(nd)}{n},$$

by sampling the process $Y(t)$ with sample distance $d$. There are two possible choices of sample distances, $d = 1/4$ and $d = 1/2$.

a) Calculate the covariance function $r_Y(\tau)$.

(4p)

b) Determine the spectral density of the sampled process $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \dots$ for the two suggested sample distances, $d = 1/4$ and $d = 1/2$.

(10p)

c) Determine $nV[\widehat{m}_n]$, when $n \to \infty$ for $d = 1/4$ and $d = 1/2$. Which of the suggested sample distances would you prefer?

(6p)

**6)** A zero-mean weakly stationary process $X(t), t \in \mathbb{R}$, has the covariance function

$$r_X(\tau) = e^{-\tau^2/2}.$$

The measurements of $X(t)$ are disturbed by zero-mean, additive measurement noise $e(t)$, giving $Y(t) = X(t) + e(t)$. The disturbance is assumed to have the covariance function

$$r_e(\tau) = \begin{cases} \sigma^2 & \text{for} \quad |\tau| \leq h/2, \\ 0 & \text{for} \quad |\tau| > h/2. \end{cases}$$

The derivative of $X(t)$ should be approximated using

$$Z(t) = \frac{aY(t) - bY(t-h)}{h}.$$

a) Calculate an expression for the variance $V[Z(t)]$.

(5p)

b) Find an expression for the cross-covariance $r_{X',Z}(\tau) = C[X'(t), Z(t+\tau)]$ where $X'(t)$ is the true derivative of $X(t)$.

(5p)

c) Minimize the mean squared error $E[(X'(t) - Z(t))^2]$ for $\sigma^2 = 1$, $h = 0.5$ and find the corresponding optimal values of $a$ and $b$.

(10p)