1) a) The figure shows the pole-zero plots of three different discrete time models of stationary processes and modified periodogram estimates of one realization from each model. A Hanning window is used as data window and the spectral estimates are presented in dB-scale. Combine the pole-zero plot with the corresponding spectral estimate. Motivate your answer. Also state which type of model (AR, MA or ARMA) in each case and the corresponding order.

![Pole-zero plots](image)

b) We have access to 1024 samples of data, where the true spectral density function is $R_X(f)$. The Welch method, with window length 256 in each periodogram and 50% overlap between windows, is applied. Approximately how large is the variance for $0 < |f| < 0.5$? Motivate your answer.

2) An MA(q)-process $X_t$, $t = 0, \pm 1, \pm 2, \ldots$, is defined as

$$X_t = e_t + b_1e_{t-1} + b_2e_{t-2} + \ldots + b_qe_{t-q},$$

where $e_t$ is white Gaussian noise with $V[e_t] = 1$. The covariance function is $r_X(0) = 2$, $r_X(\pm 3) = 1$ and zero for all other values of $\tau$. Determine the order of the process $q$, the parameters $b_1 \ldots b_q$ and the spectral density.
3) A zero-mean, weakly stationary, Gaussian process $X(t), t \in \mathbb{R}$, has the spectral density

$$ R_X(f) = \pi e^{-2\pi|f|}. $$

State, with a short motivation, for each of the following statements if it is right or wrong. Each correct answer with correct motivation gives 2 credits.

a) The covariance function is

$$ r_X(\tau) = \frac{1}{1 + \tau^2}. $$

b) The process is differentiable in quadratic mean.

c) The variance

$$ V[\frac{(X(0) + X(t))}{2}] $$

approaches zero when $t \to \infty$.

d) After filtering through a linear filter with frequency function

$$ H(f) = 1 + if $$

the spectral density of the process is constant for all frequencies.

e) A new process

$$ Y(t) = 3X(t) - 2X'(t) $$

is created. The process $Y(t)$ is a Gaussian process.

(10p)

4) For an AR(2)-process $X_t, t = 0, \pm 1, \pm 2, \ldots$, the following covariance values are given,

$$ r_X(\tau) = \begin{cases} 12 & \tau = 0, \\ 8 & |\tau| = 1, \\ 2 & |\tau| = 2, \\ -2 & |\tau| = 3. \end{cases} $$

a) Determine the spectral density of the AR(2)-process. For full number of credits, a real-valued expression should be presented.

(14p)

b) Compute the frequency function of a Wiener filter which reconstructs the AR(2)-process $X_t$ from the measurement data $Y_t = X_t + N_t$, where $N_t, t = 0, \pm 1, \pm 2, \ldots$, is a Gaussian white noise sequence with $E[N_t] = 0$ and $V[N_t] = 4$. Note that the expression should be simplified as far as possible and written in real-valued form.

(6p)
5) We define the weakly stationary Gaussian process \( Y(t) = m + X(t) \), \( t \in \mathbb{R} \), where \( m \) is an unknown constant expected value. The process \( X(t) \) is assumed to have \( E[X(t)] = 0 \) and spectral density

\[
R_X(f) = \begin{cases} 
1 & \text{for } |f - 2.1| \leq 0.1, \\
1 & \text{for } |f + 2.1| \leq 0.1 \\
0 & \text{otherwise}.
\end{cases}
\]

An estimate of \( m \) should be found as

\[
\hat{m}_n = \frac{Y(d) + Y(2d) + \ldots Y(nd)}{n},
\]

by sampling the process \( Y(t) \) with sample distance \( d \). There are two possible choices of sample distances, \( d = 1/4 \) and \( d = 1/2 \).

a) Calculate the covariance function \( r_Y(\tau) \).

b) Determine the spectral density of the sampled process \( Z_t = Y(t) \), \( t = 0, \pm d, \pm 2d, \ldots \) for the two suggested sample distances, \( d = 1/4 \) and \( d = 1/2 \).

c) Determine \( nV[\hat{m}_n] \), when \( n \to \infty \) for \( d = 1/4 \) and \( d = 1/2 \). Which of the suggested sample distances would you prefer?

6) A zero-mean weakly stationary process \( X(t), t \in \mathbb{R} \), has the covariance function

\[ r_X(\tau) = e^{-\tau^2/2}. \]

The measurements of \( X(t) \) are disturbed by zero-mean, additive measurement noise \( e(t) \), giving \( Y(t) = X(t) + e(t) \). The disturbance is assumed to have the covariance function

\[
r_e(\tau) = \begin{cases} 
\sigma^2 & \text{for } |\tau| \leq h/2, \\
0 & \text{for } |\tau| > h/2.
\end{cases}
\]

The derivative of \( X(t) \) should be approximated using

\[ Z(t) = \frac{aY(t) - bY(t - h)}{h}. \]

a) Calculate an expression for the variance \( V[Z(t)] \).

b) Find an expression for the cross-covariance \( r_{X',Z}(\tau) = C[X'(t), Z(t + \tau)] \) where \( X'(t) \) is the true derivative of \( X(t) \).

c) Minimize the mean squared error \( E[(X'(t) - Z(t))^2] \) for \( \sigma^2 = 1 \), \( h = 0.5 \) and find the corresponding optimal values of \( a \) and \( b \).