1) The pole-zero plots and the spectral densities can be combined as A-III, B-II and C-I. The strongest peak in I has a somewhat higher frequency than the other spectral densities which connects to the covariance function in 1. The spectral density in I is also more damped than the strongest peaks in II and III which also corresponds to the more damped covariance function in 1. The spectral density in III has a stronger peak than the highest in II and connects to the more undamped covariance function in 2. The spectral density in II also has a peak at a larger frequency value which shows up as the non-smooth shape of the covariance function in 3. A-III-2 is AR(3), B-II-3 is AR(4) and C-I-1 is AR(2).

2) The expected value is given from
\[ E[X_t] = 0.6E[e_t] + 0.3E[e_{t-1}] + 0.1E[e_{t-2}] = 0.6m + 0.3m + 0.1m = m. \]
The covariance function is
\[ r_X(\tau) = C[X(t), X(t + \tau)], \]
\[
\begin{align*}
  r_X(0) &= (0.6^2 + 0.3^2 + 0.1^2)\sigma^2 = 0.46\sigma^2 \\
  r_X(1) &= (0.6 \cdot 0.3 + 0.3 \cdot 0.1)\sigma^2 = 0.21\sigma^2 \\
  r_X(2) &= (0.6 \cdot 0.1)\sigma^2 = 0.06\sigma^2,
\end{align*}
\]
and \( r_X(\tau) = 0 \) for \( \tau > 2 \). Symmetry gives \( r_X(-\tau) = r_X(\tau) \). The spectral density is
\[ R_X(f) = \sum_{\tau=-\infty}^{\infty} e^{-2\pi i f \tau} r_X(\tau) = \sigma^2(0.46 + 0.42 \cos(2\pi f) + 0.12 \cos(4\pi f)). \]

3) The variance of \( \hat{\beta}_1 \) is
\[ V[\hat{\beta}_1] = V[\frac{X_5 - X_1}{4}] \]
\[
= \frac{1}{16}(r_X(0) - 2r_X(4) + r_X(0)) \\
= \frac{1}{16}(2 - 2e^{-4/2}) = 0.1081,
\]
and the variance of \( \hat{\beta}_2 \) is
\[ V[\hat{\beta}_2] = V[\frac{2X_5 + X_4 - X_2 - 2X_1}{4 + 1 + 1 + 4}] \]
\[
= \frac{1}{100}(10r_X(0) + 8r_X(1) - 2r_X(2) - 8r_X(3) - 8r_X(4)) = 0.1125.
\]
Accordingly is \( \hat{\beta}_1 \) more reliable as \( V[\hat{\beta}_1] < V[\hat{\beta}_2] \).
4) a) The Yule-Walker equations are

\[
\begin{align*}
    r_X(0) + r_X(1) + 0.5r_X(2) &= 1 \\
    r_X(1) + r_X(0) + 0.5r_X(1) &= 0 \\
    r_X(2) + r_X(1) + 0.5r_X(0) &= 0,
\end{align*}
\]

with solution \( r_X(0) = 12/5, r_X(1) = -8/5 \) and \( r_X(2) = 2/5 \). From the next step of the Yule-Walker equations, \( r_X(3) + r_X(2) + 0.5r_X(1) = 0, r_X(3) = 2/5 \). Due to symmetry we get \( r_X(0) = 12/5, r_X(1) = -8/5, r_X(2) = 2/5 \) and \( r_X(3) = 2/5 \).

b) The characteristic equation of the AR(2)-polynomial is \( z^2 + z - 0.5 = 0 \) which has solutions \( z_1 \approx 0.37 \) and \( z_2 \approx -1.37 \). The variances are \( V[Y_2] = V[e_2] = 1, V[Y_3] = V[e_2 + e_3] = 2 \) and \( V[Y_4] = V[1.5e_2 - e_3 + e_4] = 4.25 \). The pole \( |z_2| > 1 \) is outside the unit circle, which causes an unstable process. The variances increase for larger values of \( t \) and will continue to do so as more and more independent noise variables are included in the variances for larger \( t \).

5) a) The covariance function is two times differentiable as

\[
\begin{align*}
    r_X(\tau) &= e^{-\alpha \tau^2/2}, \\
    r'_X(\tau) &= -\alpha \tau e^{-\alpha \tau^2/2}, \\
    r''_X(\tau) &= (\alpha^2 \tau^2 - \alpha) e^{-\alpha \tau^2/2} = -r_X'(\tau).
\end{align*}
\]

b) For a Gaussian process, all linear combinations of the process values have a Gaussian distribution. For the Gaussian process,

\[
Y(t) = X'(t) - \frac{X(t + 0.1) - X(t)}{0.1},
\]

the expected value is

\[
E[Y(t)] = E[X'(t)] - E[X(t + 0.1)] - E[X(t)] = 0 - \frac{m - m}{0.1} = 0.
\]

as the derivative \( X'(t) \) always has the expected value zero. The variance is given as

\[
V[Y(t)] = C \left[ X'(t) - \frac{X(t + 0.1) - X(t)}{0.1}, X'(t) - \frac{X(t + 0.1) - X(t)}{0.1} \right]
\]

\[
= -r'_X(0) + 10r'_X(0.1) + 10r'_X(0) \\
-10r'_X(-0.1) + 100r_X(0) - 100r_X(0.1) \\
-10r'_X(0) - 100r_X(-0.1) + 100r_X(0) \\
= \frac{2}{0.1^2}(1 - 0.9) - 200\ln 0.9 - 2(-200\ln 0.9){0.9} \approx 3.1423.
\]

Then

\[
P(Y(t) > 0.1) = 1 - P(Y(t) \leq 1) = 1 - \Phi((1 - 0)/\sqrt{3.14}) = 1 - \Phi(0.5641) = 0.29.
\]
a) The spectral densities are given as

\[ R_S(f) = \sum_\tau e^{-i2\pi f \tau} r_X(\tau) = 4 - \left(e^{i2\pi f} + e^{-i2\pi f}\right) = 4 - 2\cos 2\pi f, \]

\[ R_N(f) = \sum_\tau e^{-i2\pi f \tau} r_N(\tau) = 4 + \left(e^{i2\pi f} + e^{-i2\pi f}\right) = 4 + 2\cos 2\pi f, \]

and the frequency function of the Wiener filter is

\[ H(f) = \frac{R_S(f)}{R_S(f) + R_N(f)} = \frac{1}{2} - \frac{1}{4} \cos 2\pi f = \frac{1}{2} - \frac{1}{8} \left(e^{-i2\pi f} + e^{i2\pi f}\right). \]

The frequency function of the specified filter structure \( Y_t = aX_{t-1} + bX_t + cX_{t+1} \) is \( H(f) = \sum_u e^{-i2\pi fu} h_u = ae^{-i2\pi f} + b + ce^{i2\pi f} \). Identification of coefficients gives \( a = c = -1/8 \) and \( b = 1/2 \).

b) With the signal and noise uncorrelated,

\[ E[(S_t - (aX_{t-1} + bX_t + cX_{t+1}))^2] = \\
4((1 - b - 0.1c)^2 + (a + 0.1b)^2 + (0.1a)^2 + c^2) \\
-2(1 - b - 0.1c)(-a - 0.1b) + 2c(1 - b - 0.1c) \\
+0.2a(-a - 0.1b) + 4(a^2 + b^2 + c^2) + 2ab + 2bc. \]

The minimum is found from the derivatives with respect to \( a, b, c \) set to zero, which gives the following system of equations

\[
\begin{align*}
15.68a - 0.78b - 0.2c &= -2 \\
0.78a + 15.68b + 0.78c &= 7.8 \\
-0.2a + 0.78b + 15.68c &= 0.8,
\end{align*}
\]

with the solution \( a = -0.10, b = 0.510, \) and \( c = 0.025 \).