1) Figures C, E and G are realizations: they cannot be covariance functions as the maximum is not at zero and they cannot be spectral densities as they include negative values. Figures A, B, D and I have maximum value at $\tau = 0$ and thereby they are possible covariance functions. Figures D, F and H are non-negative functions and therefore the only possible spectral densities. Therefore, A, B, and I are covariance functions and D, F and H are spectral densities.

The spectral density in H has strong peaks for frequencies around 0.25 (period 4). A realization and a covariance with periods of 4 are found as G and B. The spectral density in D has a strong peak at frequency 0.1 (period 10) which corresponds to the realization E and the covariance function in A. The covariance function I corresponds to an MA(2)-process, with corresponding spectral density represented by two zeros, which have to be F with zeros at $f = \pm 0.2$. The power is mainly found at higher frequencies which corresponds to realization C. The covariance functions A and B, both have non-zero values for $\tau > 4$ and have therefore to be AR-processes.

- MA(2)-process: C, Covariance: I, Spectral density: F.
- AR(3)-process: E, Covariance: A, Spectral density: D.

2) a) The covariance function is

$$r_X(\tau) = \frac{1}{2\pi^2 \tau^2} \sin^2(2\pi \tau) \quad \tau \neq 0,$$

and $r_X(0) = 2$.

b) The spectral density is

$$R_Y(f) = \begin{cases} 
(1 - |f|/2) & |f| \leq 1 \\
1/2 & 1 \leq |f| \leq \frac{3}{2}.
\end{cases}$$

3) The covariance functions of the two possible processes are, for alternative 1,

$$r_X(\tau) = \begin{cases} 
6 & \tau = 0, \\
4 & \tau = \pm 1, \\
1 & \tau = \pm 2,
\end{cases}$$

and zero for all other values, and for alternative 2,

$$r_X(\tau) = \begin{cases} 
6 & \tau = 0, \\
-4 & \tau = \pm 1, \\
1 & \tau = \pm 2,
\end{cases}$$

and zero for all other values. Alternative 2 is the most likely process as $\hat{r}_x(1)$ is found close to $-4$. With this assumption the variances of the two estimates $m_1^*$ and $m_2^*$ are

$$V[m_1^*] = 4/9 \quad V[m_2^*] = 7/2.$$  

Conclusion: The optimal estimator has the minimum variance $V[m_1^*] = 4/9$.

4) a) The spectral density is

$$R_X(f) = \frac{1}{|1 - \frac{5}{6} \exp(-i2\pi f) + \frac{1}{6} \exp(-i4\pi f)|^2}$$

$$= \frac{1}{31 - 35 \cos 2\pi f + \frac{1}{3} \cos 4\pi f} = \frac{18}{31 - 35 \cos 2\pi f + 6 \cos 4\pi f}.$$
b) The Yule-Walker equations yields,

\[
\begin{align*}
    r_X(0) - \frac{5}{6}r_X(1) + \frac{1}{6}r_X(2) &= 1 \\
r_X(1) - \frac{5}{6}r_X(0) + \frac{1}{6}r_X(1) &= 0 \\
r_X(2) - \frac{5}{6}r_X(1) + \frac{1}{6}r_X(0) &= 0
\end{align*}
\]

\[\iff\]

\[
\begin{align*}
    r_X(0) &= 2.1 \\
r_X(1) &= 1.5 \\
r_X(2) &= 0.9
\end{align*}
\]

using that \(r_X(-\tau) = r_X(\tau)\). For \(\tau \geq 2\), \(r_X(\tau) = \frac{5}{6}r_X(\tau - 1) - \frac{1}{6}r_X(\tau - 2)\) and the expression for the covariance function is finally

\[
r_X(\tau) = \frac{24}{5} \left( \frac{1}{2} \right)^{|\tau|} - \frac{27}{10} \left( \frac{1}{3} \right)^{|\tau|}.
\]

5) \[
V[Y] = C \left[ \int_0^\pi X(t) dt - \int_0^{2\pi} X(t) dt, \int_0^\pi X(s) ds - \int_0^{2\pi} X(s) ds \right] = \int_0^\pi \int_0^\pi r_X(t-s) dt ds - \int_0^\pi \int_0^\pi r_X(t-s) dt ds
\]

\[\iff\]

\[
\int \int \cos^2\left( t - \frac{s}{2} \right) dt ds = \frac{1}{2} \int \int (1 + \cos(t-s)) dt ds,
\]

\[= \frac{1}{2} \int \int (1 + \cos(t) \cos(s) + \sin(t) \sin(s)) dt ds,
\]

\[= \frac{1}{2} \pi^2 + \frac{1}{2} \int \cos(t) dt \int \cos(s) ds + \frac{1}{2} \int \sin(t) dt \int \sin(s) ds.
\]

The variance becomes

\[
V[Y] = \left( \frac{\pi^2}{2} + 2 \right) - \left( \frac{\pi^2}{2} - 2 \right) - \left( \frac{\pi^2}{2} - 2 \right) + \left( \frac{\pi^2}{2} + 2 \right) = 8.
\]

6) With \(\epsilon_{t+1} = X_{t+1} - \hat{X}_{t+1}\) the minimization of

\[
P(|\epsilon_{t+1}| > 1) = 1 - P(-1 < \epsilon_{t+1} < 1),
\]

where \(\epsilon_{t+1} \in N(0, V[\epsilon_{t+1}])\) is the same as minimizing the variance \(V[\epsilon_{t+1}]\). We find

\[
V[\epsilon_{t+1}] = V[\epsilon_{t+1} + e_t + e_{t-1} - a(\epsilon_t + e_{t-1} + e_{t-2}) - b(\epsilon_{t-1} + e_{t-2} + e_{t-3})],
\]

\[= \frac{1}{4}(1 + (1-a)^2 + (1-a-b)^2 + (-a-b)^2).
\]

Minimization with respect to \(a\) and \(b\) gives

\[
\frac{\delta V}{\delta a} = \frac{3}{2} a + b - 1 = 0,
\]

\[
\frac{\delta V}{\delta b} = a + \frac{3}{2} b - \frac{1}{2} = 0,
\]

with the solution \(a = \frac{4}{5}\) and \(b = \frac{-1}{5}\).