1) a) The figure shows the poles and zeros of 3 different discrete time stationary stochastic processes and the corresponding modified periodogram estimates using a Hanning window, in dB-scale. Combine the pole-zero plot with the corresponding spectral estimate. Motivate your answer. Also state which type of process (AR, MA or ARMA) and the corresponding orders.

b) We have access to 256 samples of data, where the true spectral density function is \( R_X(f) \). The Welch method, with window length 64 in each periodogram and 50% overlap between windows, is applied. Approximately how large is the variance for \( 0 < |f| < 0.5 \)? Motivate your answer.

2) The average level \( m \) of a stationary stochastic process \( Y_t = X_t + m, \ t = 0, \pm 1, \pm 2 \ldots \), should be estimated. The disturbance process \( X_t \) is one of two models, namely

\[
X_t = e_t + e_{t-1},
\]

or

\[
X_t = 0.8X_{t-1} + e_t,
\]

where \( e_t, t = 0, \pm 1, \pm 2 \ldots \), is white Gaussian noise with expected value zero. From one realization, a covariance function estimate, \( \hat{r}_x(\tau) \) is computed, (see the figure below).
Determine and motivate which of the two suggested disturbance models above it is most likely to be. Compute the true covariance function of your choice and use this in the following calculations.

Two different mean value estimators are suggested,

\[ \hat{m}_1 = \frac{Y_t + Y_{t-1} + Y_{t-2}}{3}, \]

and

\[ \hat{m}_2 = \frac{Y_t + Y_{t-2}}{2}. \]

Investigate and determine which of \( \hat{m}_1 \) and \( \hat{m}_2 \) that has the lowest variance.

(10p)

3) A causal matched filter should be determined to detect the signal,

\[ s_t = \begin{cases} 
1 & t = 0, \\
2 & t = 1, \\
4 & t = 2, 
\end{cases} \]

where the received signal is disturbed by zero-mean Gaussian white noise \( N_t \) with variance \( V[N_t] = 4 \). The received observation is \( X_t = s_t + N_t \) if signal is sent or alternatively \( X_t = N_t \) if no signal is sent. The error probabilities

\[ \alpha = P(\text{detect signal if no signal is sent}) \]
\[ \beta = P(\text{detect no signal if signal is sent}) \]

should be equal and as small as possible. Determine the parameters of the causal matched filter, the decision threshold \( k \), and the error probabilities \( \alpha = \beta \).

(10p)

4) A stationary continuous time process is given as

\[ X(t) = \sum_{k=1}^{3} A_k \cos(2\pi f_k t + \phi_k), \quad -\infty < t < \infty, \]

where the stochastic variables \( A_k, \phi_k, k = 1 \ldots 3 \) all are independent and \( \phi_k \in \text{Rect}(0, 2\pi) \). The frequencies are \( f_1 = 1000 \text{ Hz}, f_2 = 5000 \text{ Hz}, \) and \( f_3 = 17000 \text{ Hz} \).

Determine if the following statements are correct or false. Note: for full number of credits the answer must include an extensive motivation.
a) The process $X(t)$ is sampled with $f_s = 7000$ Hz. The frequencies after sampling are 1000 Hz, 2000 Hz, and 3000 Hz. (4p)

b) The process $X(t)$ is filtered giving $Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$, with

$$h(t) = \begin{cases} 10000 & \text{if } t = 0, \\ \frac{1}{20000\pi^2t^2}(1 - \cos(20000\pi t)) & \text{if } t \neq 0. \end{cases}$$

Then the process $Y(t)$ is sampled with $f_s = 7000$ Hz. The resulting frequencies after sampling are 1000 Hz and 3000 Hz. (8p)

c) The process $X(t)$ is sampled with $f_s = 20000$ Hz and is then converted into the discrete time sequence $Z_t$, $t = 0, \pm 1, \pm 2, \ldots$. The sequence $Z_t$ is filtered giving $W_t = \sum_{u=-\infty}^{\infty} g(u)Z_{t-u}$ using the ideal low-pass filter $g(t)$, $t = 0, \pm 1, \pm 2, \ldots$, with corresponding frequency function,

$$G(\nu) = \begin{cases} 1 & |\nu| \leq \frac{3}{16}, \\ 0 & \frac{3}{16} < |\nu| \leq \frac{1}{2}, \end{cases}$$

where $\nu$ is normalized frequency. The sequence $W_t$, $t = 0, \pm 1, \pm 2, \ldots$, contains only two frequencies after the discrete time filtering. (8p)

5) A discrete time process $S_t$, $t = 0, \pm 1, \pm 2, \ldots$ is disturbed by noise $N_t$, $t = 0, \pm 1, \pm 2, \ldots$. To improve the measured signal, $Z_t = S_t + N_t$ is filtered in a filter with frequency function

$$H(f) = a_0 + a_1 e^{-i2\pi f},$$

giving the output process $Y_t$. Determine the coefficients $a_0$ and $a_1$ so that the mean squared error, $E[(Y_t - S_t)^2]$, becomes as small as possible. The processes $S_t$ and $N_t$ are independent and have expected values, $m_S = m_N = 0$, and covariance functions $r_S(\tau) = 0.5|\tau|$ and $r_N(\tau) = (-0.5)|\tau|$, $\tau = 0, \pm 1, \pm 2, \ldots$. (20p)

6) A step signal

$$S(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0, \end{cases}$$

is disturbed by a stationary Gaussian process $X(t)$ with expected value $m_X = 0$ and covariance function $r_X(\tau) = e^{-\tau^2}$. The process $Y(t) = S(t) + X(t)$ is filtered in a filter with impulse response

$$h(t) = e^{-t^2/2g^2}, \ -\infty < t < \infty,$$

where the output process is $U(t)$. In connection to edge detection it is interesting to differentiate $U(t)$. Compute $V[U'(0)]/E[U(0)^2]$ as a function of the parameter $g$. The expression of the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi},$$

might be useful. (20p)