Solutions, correct and well motivated, of exercise 1-3 give 10 credits and of exercise 4-6 give 20 credits. Maximal number of credits are 90. The limit for passed the course is 40 credits. Each exercise should have a solution starting on a new sheet of paper.

1) The covariance function always has the largest value at $\tau = 0$. The only possible alternatives are A, C, F. A spectral density is always positive, which is true for B, D, H. Thus E, G, I are realizations.

The covariance functions in A and F belong to AR-processes, as they certainly have covariance values for $\tau > 10$. A varies faster than F, which correspond to the strong spectral peak at $f = 0.5$ in D and the realization in E, where F corresponds to the spectral peak in H and the process realization in G. The orders are AR(3) for A, D, E and AR(4) for F, H, G. The covariance function in C can be an MA(4)-process as it is zero for $\tau \geq 5$ and the spectral density function is the one in B together with the noisy realization in I.

2) The Wiener filter is given from

$$H(f) = \frac{R_X(f)}{R_X(f) + R_N(f)}$$

where

$$R_X(f) = \frac{2}{1 + (2\pi f)^2},$$

and

$$R_N(f) = \frac{2\beta}{\beta^2 + (2\pi f)^2}.$$ 

We get

$$H(f) = \frac{\beta^2 + 4\pi^2 f^2}{\beta^2 + \alpha\beta + 4\pi^2(1 + \alpha\beta)f^2}.$$ 

3) The expected value of the derivative is always $E[X'(t)] = 0$ and the spectral density is $R_X'(f) = (2\pi f)^2R_X(f)$. The variance of the derivative is

$$V[X'(t)] = \int R_X'(f)df = c\int_{f_0}^{f_0} (2\pi f)^2(1 - |f|/f_0)df$$

$$= 2c\int_{f_0}^{f_0} (2\pi f)^2(1 - f/f_0)df = 8\pi^2 cf_0^3/12 = 1.$$ 

4) With one accelerometer we get $V[\hat{m}_1] = V[X_1] = 1$. With two accelerometers we get

$$V[\hat{m}_2] = \frac{1}{4}V[X_1 + X_2] = \frac{1}{4}(2r(0) + 2r(d)),$$

where $r(d)$ is minimum when the measurement distance is as large as possible. The maximum length of the cable is limited by the maximum cost, and will then be $(12000 - 2*3000)/100 = 60m$. We get

$$V[\hat{m}_2] = \frac{1}{4}(2 + 2 \cdot 0.00248) = 0.501.$$ 

With 3 accelerometers we get

$$V[\hat{m}_3] = \frac{1}{9}V[X_1 + X_2 + X_3] = \frac{1}{9}(3r(0) + 4r(d) + 2r(2d)),$$

where the maximum cable length is $(12000 - 3*3000)/100 = 30m$ giving $d = 15m$ and

$$V[\hat{m}_3] = \frac{1}{9}(3 + 4 \cdot 0.223 + 2 \cdot 0.0498) = 0.443.$$ 

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With 4 accelerometers there is no money left for connecting cable so all accelerometers should be placed at the same point giving 

\[ V[\hat{m}_4] = \frac{1}{16} V[X_1 + X_2 + X_3 + X_4] = \frac{1}{16} (4r(0) + 12r(0)) = 1. \]

Answer: A number of 3 accelerometers will give the smallest variance \( V[\hat{m}_3] = 0.443. \)

5) Yule-Walker gives

\[
\begin{align*}
    r_X(0) + a_1 \cdot 1.6 + a_2 \cdot 0.40 &= \sigma^2 \\
    1.6 + a_1 \cdot r_X(0) + a_2 \cdot 1.6 &= 0 \\
    0.4 + a_1 \cdot 1.6 + a_2 \cdot r_X(0) &= 0 \\
    -0.4 + a_1 \cdot 0.4 + a_2 \cdot 1.6 &= 0
\end{align*}
\]

solved by \( a_2 = 1/4 - 1/4a_1, r_X(0) = -\frac{1+a_2}{a_1} \cdot 1.6 \) and \( a_1^2 + 2/3 \cdot a_1 - 1/3 = 0 \) with two \( a_1 = -1 \) and \( a_1 = 1/3 \). We have the two following alternatives: Alternative 1 with \( a_1 = 1/3, a_2 = 1/6 \) gives \( r_X(0) = -5.6 \) which is not a variance. Alternative 2 with \( a_1 = -1, a_2 = 1/2 \) gives \( r_X(0) = 2.4 \) and \( \sigma^2 = 1 \) which is the solution.

6) With \( X_t \) as the stock price deviation we get

\[
\begin{align*}
    E \left[(X_{t+1} - \hat{X}_{t+1})^2\right] &= E \left[(X_{t+1} - (X_t + X_{t-1} + cX_{t-2})/3)^2\right] \\
    &= r_X(0)(1 + 1/9 + 1/9 + c^2/9) + 2r_X(1)(-1/3 + 1/9 + c/9) \\
    & \quad + 2r_X(2)(-1/3 + c/9) + 2r_X(3)(-c/3) \\
    &= \frac{1}{9} \left(\ldots + c^2 \cdot r_X(0) + c \cdot (2r_X(1) + 2r_X(2) - 6r_X(3))\right) \\
    &= \frac{0.028}{9 \cdot 16} \left(\ldots + 16c^2 - 20c\right),
\end{align*}
\]

with minimum \( c = 5/8. \)