1) C, G, H are spectral densities, as they are the only ones that are positive. The scales on the x-axes are given as 0-0.5 which also indicates that these may be the spectral densities.

A, E, I are covariance functions based on that the largest value of the function is found at zero, \( r_X(0) \geq r_X(\tau) \).

B, D, F are realizations.

The spectral density G belongs to realizations B that is slowly varying and thereby also must be connected to the slowly varying covariance function A.

The spectral density C belongs to the rapidly varying realization D and a rapidly varying covariance function I.

The spectral density H belongs to a process that neither is slowly or rapidly varying, F, and a corresponding covariance function E.

From the study of the covariance functions, it reasonable to believe that E is finite, i.e., an MA-process and that the order probably is MA(2) as the third and higher values are zero. A

2) The process is described by \( X_t + 0 \cdot X_{t-1} - 0.5 \cdot X_{t-2} = \sigma^2 \), yielding the Yule-Walker equations:

\[
\begin{align*}
    r(0) + a_1r(1) + a_2r(2) &= \sigma^2, \\
    r(1) + a_1r(0) + a_2r(1) &= 0, \\
    r(2) + a_1r(1) + a_2r(0) &= 0, \\
    r(k) + a_1r(k-1) + a_2r(k-2) &= 0 \text{ for } k = 3, 4, \ldots.
\end{align*}
\]

With \( a_1 = 0 \) and \( a_2 = -0.5 \), the second equation gives that \( r(1) = 0 \). The first and third equation gives with \( \sigma^2 = 1 \), that \( r(0) = \sigma^2/(1 - 0.5^2) = 4/3 \). From the last equation, we get \( r(k) = 0.5r(k-2) \). With \( r(0) = 4/3 \) and even values \( k = 2j \), \( r(2j) = 0.5^j r(0) \); and for odd values, \( k = 2j + 1 \), \( r(2j + 1) = 0 \), for \( j = 1, 2, 3, \ldots \). Finally, \( r(-k) = r(k) \). As a result the spectral density is

\[
R_X(f) = \frac{1}{|1 - 0.5e^{-4\pi f}|^2} = \frac{1}{1.25 - \cos(4\pi f)}.
\]

3) a) With sampling, \( t = nd \),

\[
x(n) = A_1 \cos(2\pi(\frac{4.5}{5}n + \phi_1)) + A_2 \cos(2\pi(\frac{7}{5}n + \phi_2)) + A_3 \cos(2\pi(\frac{11}{5}n + \phi_3))
\]

The sampled process consists of the frequencies \(|1 - 4.5/5| = 1/10\), \(|2 - 11/5| = 1/5\) and \(|1 - 7/5| = 2/5\). (or 500 Hz, 1 kHz, 2kHz).

b) The impulse response is

\[
h(n) = \frac{1}{2} \sin(\frac{n}{2}) = \frac{1}{2} \sin(\frac{\pi n}{2n})
\]

with frequency function

\[
H(f) = \begin{cases} 1, & |f| \leq \frac{1}{4} \\ 0, & \text{för övrigt,} \end{cases}
\]

resulting in that \( f_1 = 1/10 < 1/4 \) and \( f_2 = 1/5 < 1/4 \) are passed through the filter.
a) In the triangle, every corner has two neighbors at the distance $L_T$. In the square every corner has two neighbors at the distance $L_K$ and one neighbor at the distance $\sqrt{2} L_T$. The variances are

\[
V(X_T) = \frac{1}{9} V(X_1 + X_2 + X_3) = \frac{1}{9} (3r(0) + 3 \cdot 2 r(L_T)) = (1 + 2 e^{-L_T})/3
\]

\[
V(Y_K) = \frac{1}{16} V(Y_1 + Y_2 + Y_3 + Y_4) = \frac{1}{16} (4r(0) + 4 \cdot 2 r(L_K) + 4 r(\sqrt{2} L_K))
\]

\[= (1 + 2 e^{-L_K} + e^{-\sqrt{2} L_K})/4.\]

b) If the walk should be at most 1 km for one lap, the triangle side is $L_T = 1/3$ and the square side $L_K = 1/4$. The variances are

\[
V(X_T) = (1 + 2 e^{-1/3})/3 = 0.81102, \quad V(Y_K) = (1 + 2 e^{-1/4} + e^{-\sqrt{2}/4})/4 = 0.81495,
\]
i.e., the triangle gives the smallest variance.

c) If the walk should be at most 4 km, $L_T = 4/3$ and $L_K = 1$ giving the variances

\[
V(X_T) = (1 + 2 e^{-4/3})/3 = 0.50906, \quad V(Y_K) = (1 + 2 e^{-1} + e^{-\sqrt{2}})/4 = 0.49472,
\]
i.e., the square gives the smallest.

d) The variance for the average is smaller, the more measurements one can include, and is also decreased with the independence of the measurements. When the square and the triangle are small, the dependence in the measurements are large. If the distance can be increased (as for the triangle), the smallest variance is achieved. For the larger square and triangle, the measurements are almost independent, and the smallest variance is given for the square, which includes more measurements in the average.

5) For the output process $Y(t) = \int h(u) X(t-u) du$, the spectral density is $R_Y(f) = |H(f)|^2 R_X(f)$ where $H(f) = \int \exp(-i2\pi fu) h(u) du$. From the table of formulas $R_X(f) = \pi e^{-2\pi|f|}$ and $H(f) = (\pi/2)e^{-4\pi|f|}$, giving $|H(f)|^2 = (\pi/2)^2 e^{-8\pi|f|}$. Therefore

\[
R_Y(f) = \frac{\pi^3}{4} e^{-10\pi|f|} = 5(\pi/2)^2 \frac{\pi}{5} e^{-2\pi.5|f|}.
\]

The covariance function is

\[
r_Y(t) = 5(\pi/2)^2 \frac{1}{25 + f^2}, \quad V(Y(t)) = r_Y(0) = \pi^2/20.
\]

A process is differentiable is the spectral density is integrable, i.e., $\int (2\pi f)^2 R(f) df < \infty$. The input process $X(t)$ as well as the output process $Y(t)$ have integrable spectral densities.

6) a) $r_U(\tau) = \text{Cov}(U_t, U_{t+\tau}) = \ldots =

\[
\frac{1}{9} (3r_X(\tau) + 2r_X(\tau + 1) + 2r_X(\tau - 1) + r_X(\tau + 2) + r_X(\tau - 2))
\]

As $U$ is a stationary process, $\text{Cov}(U_t, U_{t+\tau}) = \text{Cov}(U_{t-1}, U_{t-1+\tau}) = \text{Cov}(V_t, V_{t+\tau})$ är $r_V(\tau) = r_U(\tau)$.

b) Calculate

\[
r_{XY}(\tau) = \text{Cov}(X_t, Y_{t+\tau}) = \text{Cov}(X_t, X_{t+\tau} - \frac{1}{3}(X_{t+1+\tau} + X_{t+\tau} + X_{t-1+\tau}))
\]

\[= \frac{2}{3} r_X(\tau) - \frac{1}{3} r_X(\tau - 1) - \frac{1}{3} r_X(\tau + 1).
\]
The frequency function is
\[ H_{XY}(f) = \sum_{u=\{-1,0,1\}} \exp(-i2\pi fu)h_u = \ldots = \frac{2}{3}(1 - \cos(2\pi f)). \]

For the stationary process \( Z \) similar calculations give
\[ r_{XZ}(\tau) = \frac{2}{3}r_X(\tau) - \frac{1}{3}r_X(\tau - 1) - \frac{1}{3}r_X(\tau - 2), \]
and the frequency function
\[ H_{XZ}(f) = \sum_{u=\{0,1,2\}} \exp(-i2\pi fu)h_u = \ldots = \frac{2}{3} - \frac{1}{3}\exp(-i2\pi f) - \frac{1}{3}\exp(-i4\pi f). \]

The answer is \( R_{XY}(f) = H_{XY}(f)R_X(f) \), where \( R_X(f) \) is the sum of the spectral density from \( 2^{-|\tau|} \) and the covariance function of the MA-part, giving
\[ R_X(f) = \frac{3/4}{5/4 - \cos(2\pi f)} + 1 - \cos(2\pi f) + \frac{1}{3}\cos(4\pi f). \]

\( c) \) The slow variations should reduced by using a high-pass filter. The solution is given by studying
\[ |H_{XY}(f)|^2 = \frac{4}{9}(1 - 2\cos(2\pi f) + \cos^2(2\pi f)), \]
\[ |H_{XZ}(f)|^2 = \frac{2}{9}(5 - \cos(2\pi f) - 4\cos^2(2\pi f)), \]
for \( 0 \leq f < 1/2 \), giving that \( \cos(2\pi f) \geq 0 \). Thereby \( |H_{XY}(f)| \) is larger than \( |H_{XZ}(f)| \) for \( f \) close to \( 1/2 \), and \( Y \) will include more high frequencies than \( Z \).