1) The figures below belong to processes of the type AR or MA. State, with motivation, which of the figures that are realizations, covariance functions and spectral densities. Determine which realizations, covariance functions and spectral densities that belong to the same process. Also determine the model structure (AR or MA) and suggest an appropriate model order of the processes. NOTE! To receive full credits correct motivations have to be given for all answers.

![Figures A to I](image)

2) An AR(2)-process is described by $X_t = 0.5 X_{t-2} + e_t$, where the innovation variance $V(e_t) = 1$. Determine the covariance function $r_X(\tau)$ and the spectral density $R_X(f)$. 

(10p)
3) A time-continuous process consists of a sum of three cosine signals

\[ X(t) = \sum_{k=1}^{3} A_k \cos(2\pi f_k t + \phi_k), \quad -\infty < t < \infty \]

where \( E[A_1^2] = 1, E[A_2^2] = 4, E[A_3^2] = 1 \) and the phase functions are uniformly distributed in the interval \([−\pi, \pi]\). The frequencies are \( f_1 = 4.5 \), \( f_2 = 7 \) and \( f_3 = 11 \) kHz. The time-continuous process is sampled. By mistake the sampling frequency is chosen as \( f_s = 1/d = 5 \) kHz.

a) Which frequencies are present in the sampled signal? (4p)

b) The time-discrete signal is after sampling filtered with a filter with impulse response \( h(n) = \frac{1}{2} \text{sinc}(n/2) \). Which frequencies are present in the output signal after the time-discrete filtering? (Hint: \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \).) (6p)

4) We would like to estimate the average level of a pollution of a surface with use of the average value at a number of measurement points. The following alternatives are possible:

Triangle: Use three measurements at the corners of a triangle with side lengths \( L_T \).

Square: Use four measurements at the corners of a square with side lengths \( L_K \).

We suppose that the covariance is \( e^{-d} \), where \( d \) is the distance in kilometers.

a) Write down the expression for the variance of the estimate of the average level of the two cases as functions of \( L_T \) and \( L_K \), respectively.

b) When the measurements are collected, we walk along the sides of the triangle and the square, respectively. Which of the methods give the smallest variance if we just are able to walk 1 kilometer, i.e., if the total distance around the triangle is 1 km and around the square 1 km?

c) If we instead are able to walk 4 kilometers, i.e., if the distance around the triangle is 4 km and around the square 4 km, which of the methods give the smallest variance?

d) Motivate your answer in exercise b) and c). (20p)

5) Determine the spectral density \( R_Y(f) \), the covariance function \( r_Y(\tau) \) and the variance \( V(Y(t)) \) if

\[ Y(t) = \int_{-\infty}^{\infty} h(u) X(t - u) \, du \]

when \( X(t) \) is a time-continuous stationary process with expected value zero, the covariance function \( r_X(\tau) = 1/(1 + \tau^2) \) and the impulse response \( h(t) = 1/(4 + t^2) \). (20p)
The following exercise is a simplified illustration to the problem of separating a slow and fast variation in a time series using a moving average. Let \( \{X_t, t = 0, \pm 1, \pm 2 \ldots\} \) be a weakly stationary process with covariance function

\[
r_X(\tau) = \begin{cases} 
2^{-|\tau|} + 1 & \text{for } \tau = 0 \\
2^{-|\tau|} - 1/2 & \text{for } \tau = \pm 1 \\
2^{-|\tau|} + 1/6 & \text{for } \tau = \pm 2 \\
2^{-|\tau|} & \text{for } |\tau| \geq 3
\end{cases}
\]

This means that \( \{X_t\} \) is a sum of a process with the covariance function \( 2^{-|\tau|} \) and a certain MA(3)-process.

Here are two alternatives to separate them: For every \( t \) the averages

\[
U_t = \frac{X_{t+1} + X_t + X_{t-1}}{3}, \\
V_t = \frac{X_t + X_{t-1} + X_{t-2}}{3},
\]

are calculated. The differences \( Y_t = X_t - U_t \) and \( Z_t = X_t - V_t \) are also made. The idea is that \( \{U_t\} \) and \( \{V_t\} \) should represent the slow variation of the process \( \{X_t\} \) but \( \{Y_t\} \) and \( \{Z_t\} \) should represent faster variations.

a) Compute and compare the covariance functions \( r_U(\tau) \) and \( r_V(\tau) \).

\( (6p) \)

b) Calculate the covariance functions \( r_{X,Y}(\tau) \) and \( r_{X,Z}(\tau) \) and the corresponding spectral densities \( R_{X,Y}(f) \) and \( R_{X,Z}(f) \). Hint: It is convenient to use frequency functions.

\( (12p) \)

c) Use the results in (b) to explain why it is better to use \( \{U_t\} \) and \( \{Y_t\} \) than using \( \{V_t\} \) and \( \{Z_t\} \) in the separation.

\( (2p) \)