Introduction to time-frequency analysis

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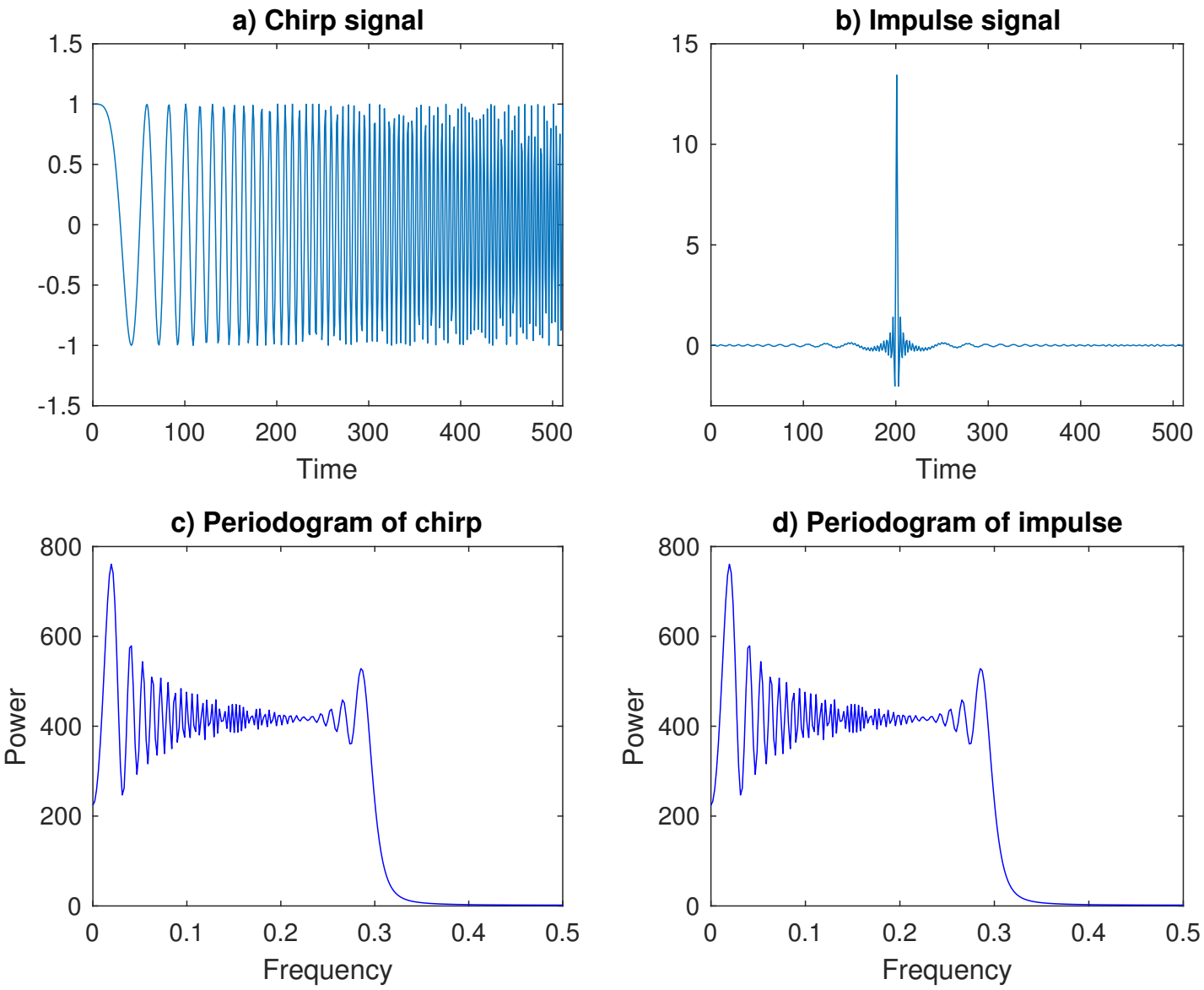
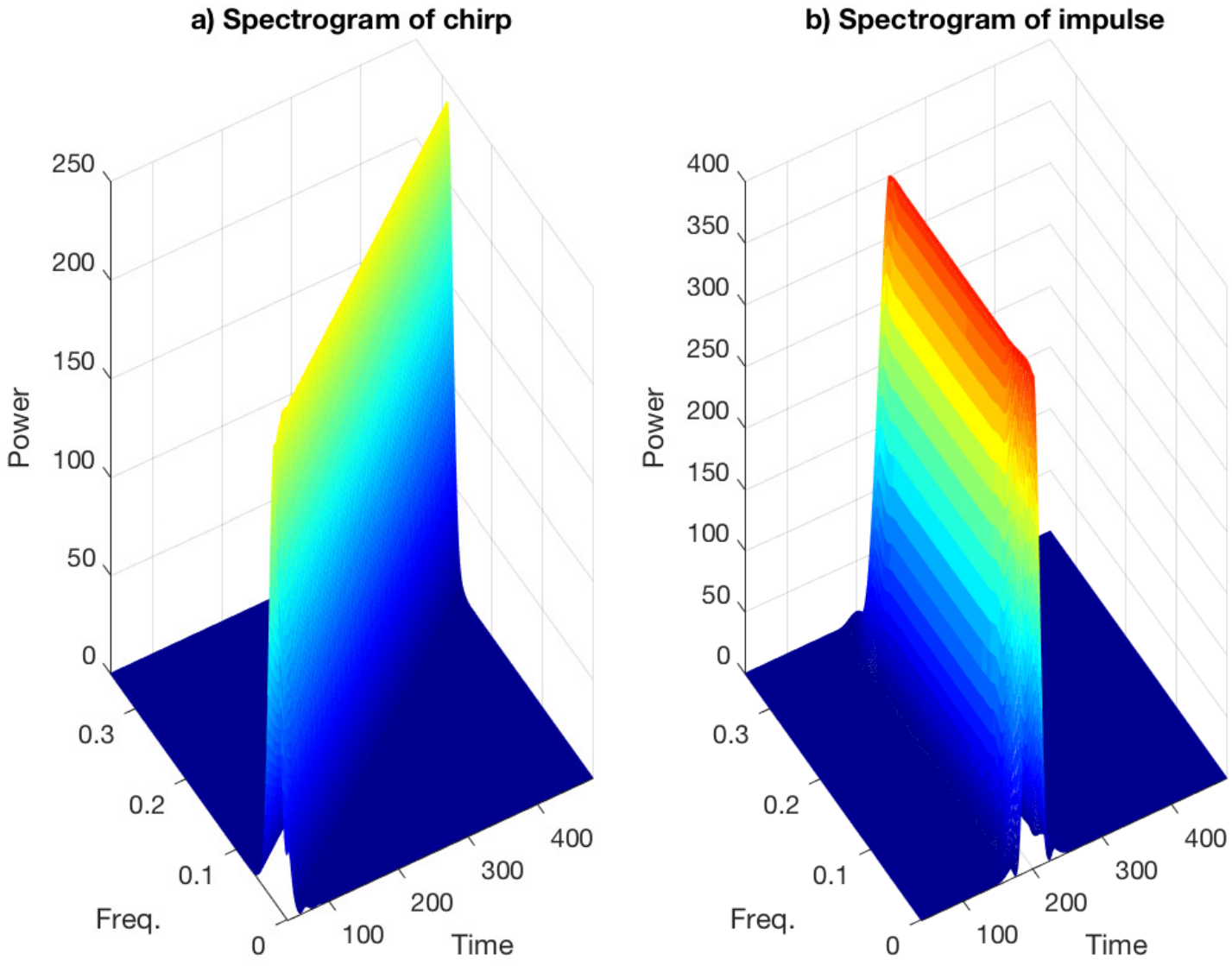
Lecture 5
Stationary and non-stationary spectral analysis

February 3 2020

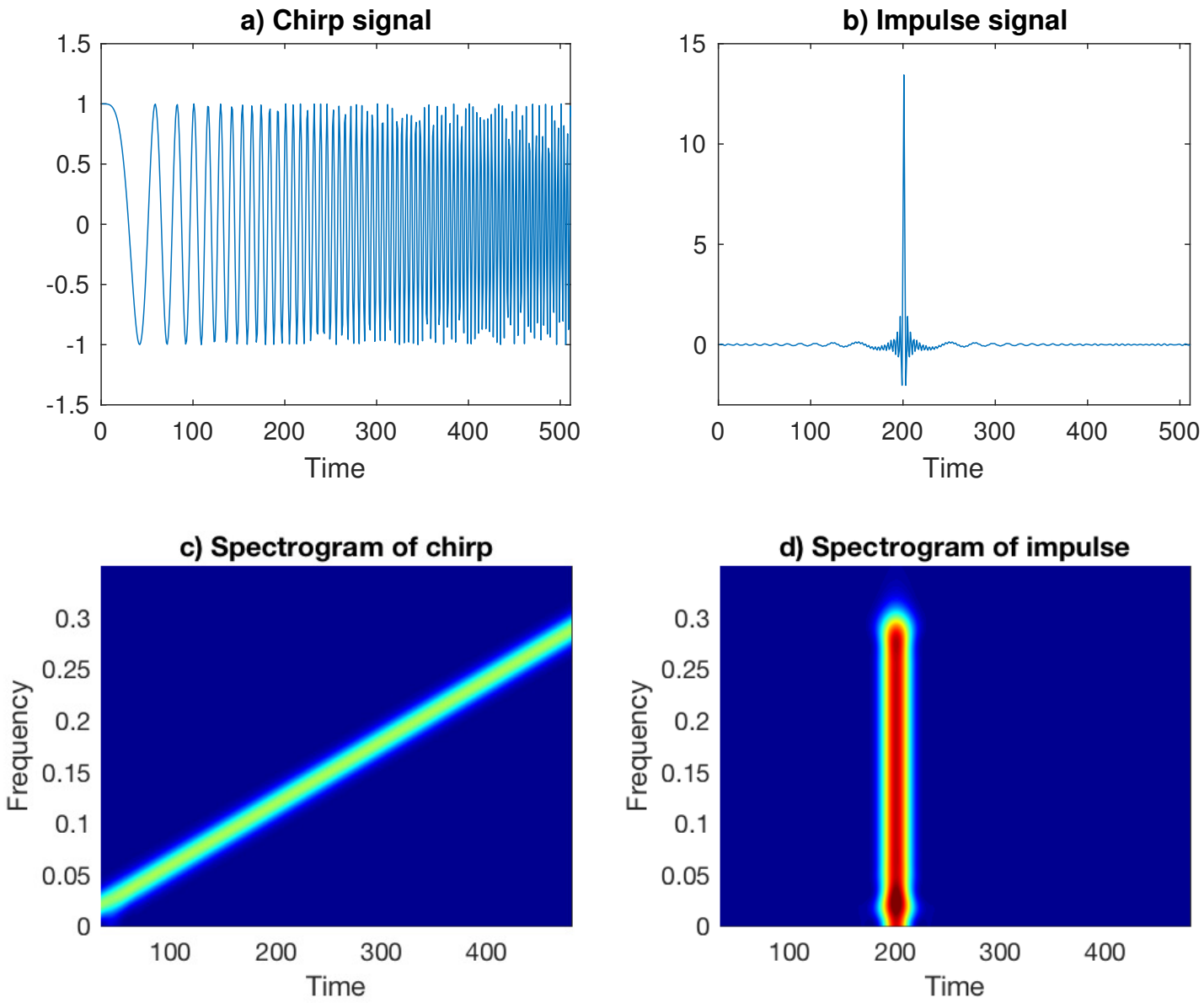
The need for time-frequency analysis



The need for time-frequency analysis



The need for time-frequency analysis



Effective duration and bandwidth

If the signal as well as the Fourier transform are assumed to be located symmetrically around $t = 0$ and $f = 0$, the **effective duration** is defined as

$$T_e = \sqrt{\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}}.$$

and the corresponding **effective bandwidth** as,

$$B_e = \sqrt{\frac{\int_{-\infty}^{\infty} f^2 |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df}}.$$

Uncertainty principle

The **bandwidth-duration product** is defined as

$$B_e \cdot T_e,$$

The product is also called the **uncertainty principle** and serves as a measure of the information richness in the signal. For all possible signals,

$$B_e \cdot T_e \geq \frac{1}{4\pi},$$

where $B_e \cdot T_e = \frac{1}{4\pi}$ only for the Gaussian signal,

$$x(t) = (\frac{\alpha}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha}{2}t^2}.$$

The spectrogram

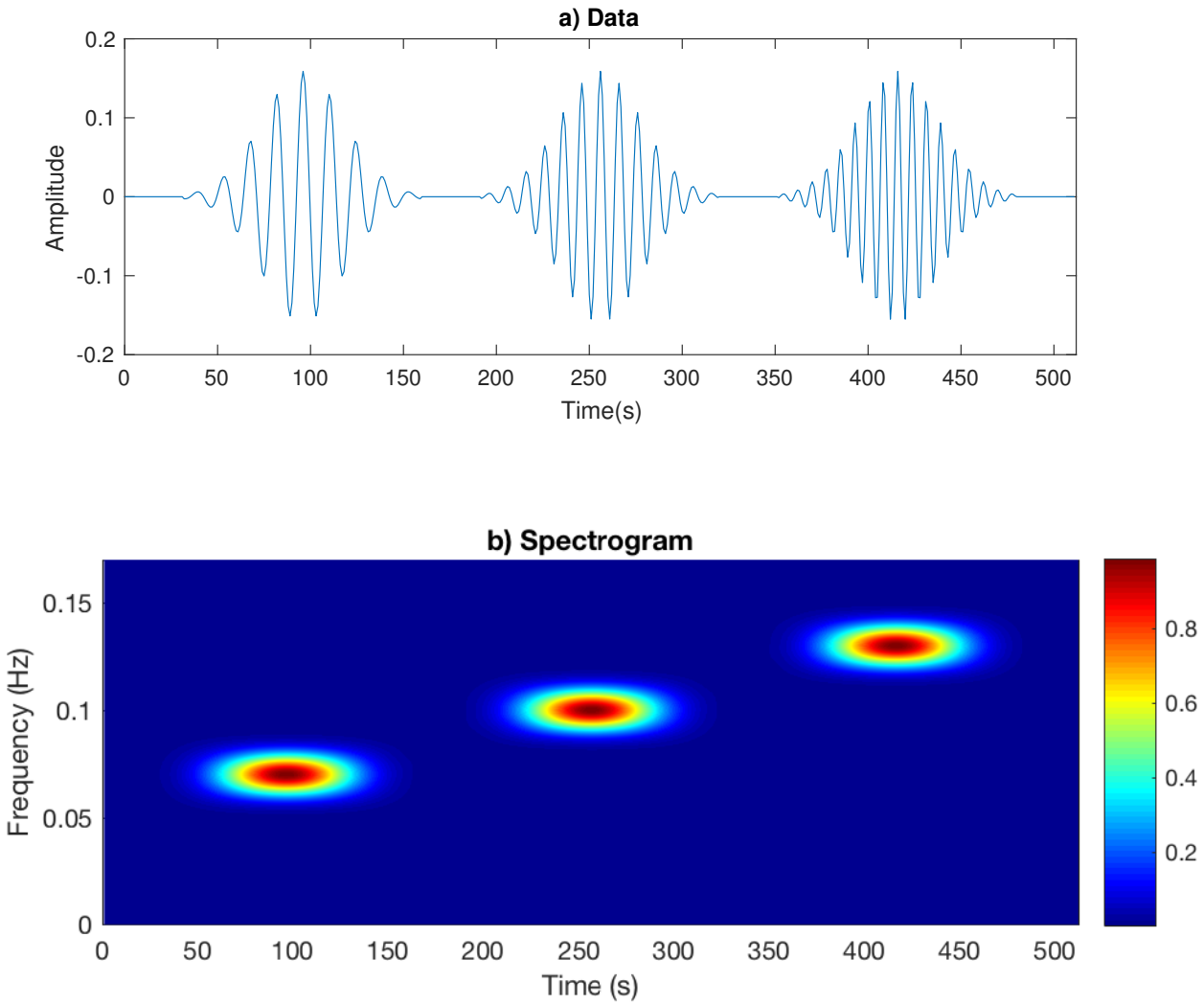
The **short-time Fourier transform (STFT)** for a signal $x(t)$ is defined as

$$X(t,f) = \int_{-\infty}^{\infty} x(t_1)h^*(t_1 - t)e^{-i2\pi f t_1} dt_1,$$

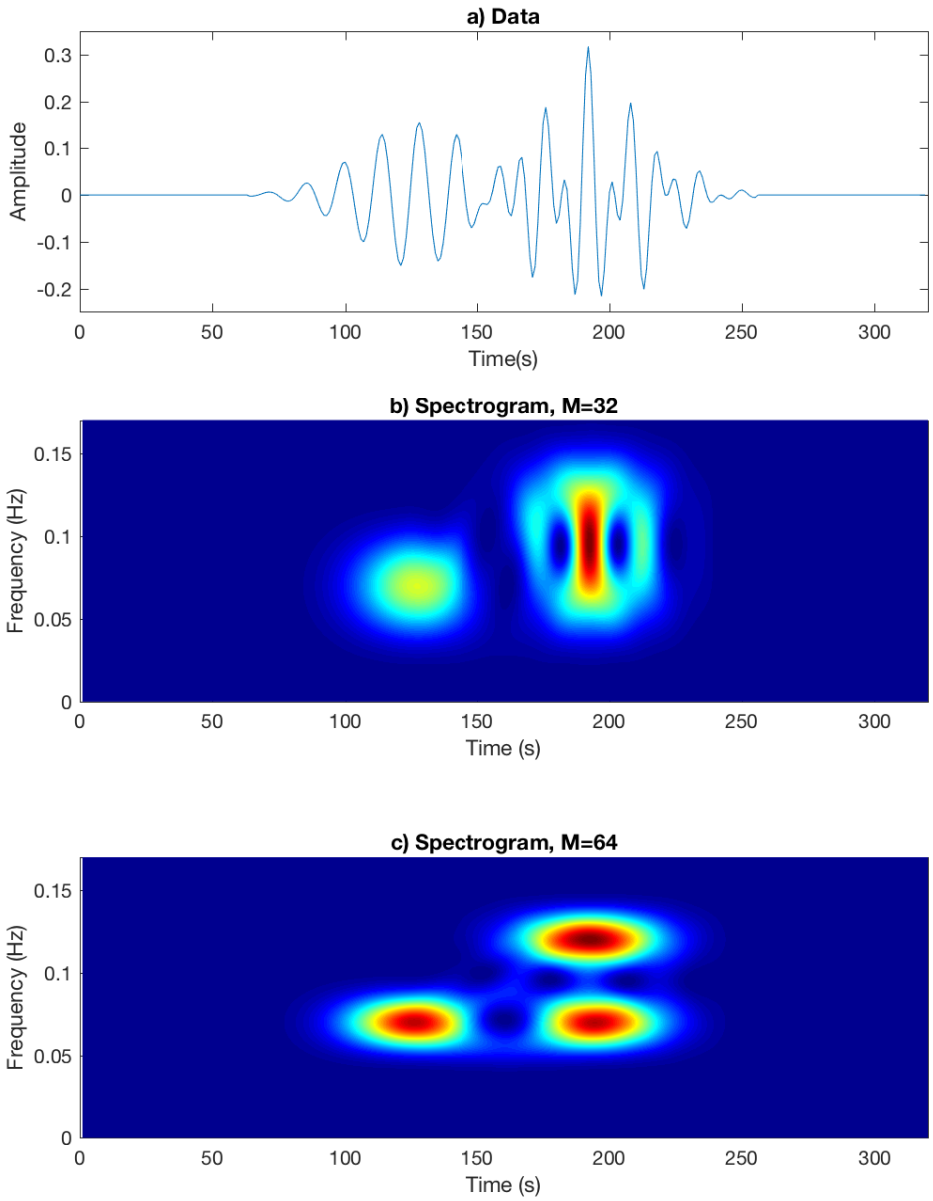
where the unit energy window function $h(t)$ centered at time t is multiplied with the signal $x(t)$ before the Fourier transform. Similar to the ordinary Fourier transform and spectrum we can formulate the **spectrogram** as

$$S_x(t,f) = |X(t,f)|^2.$$

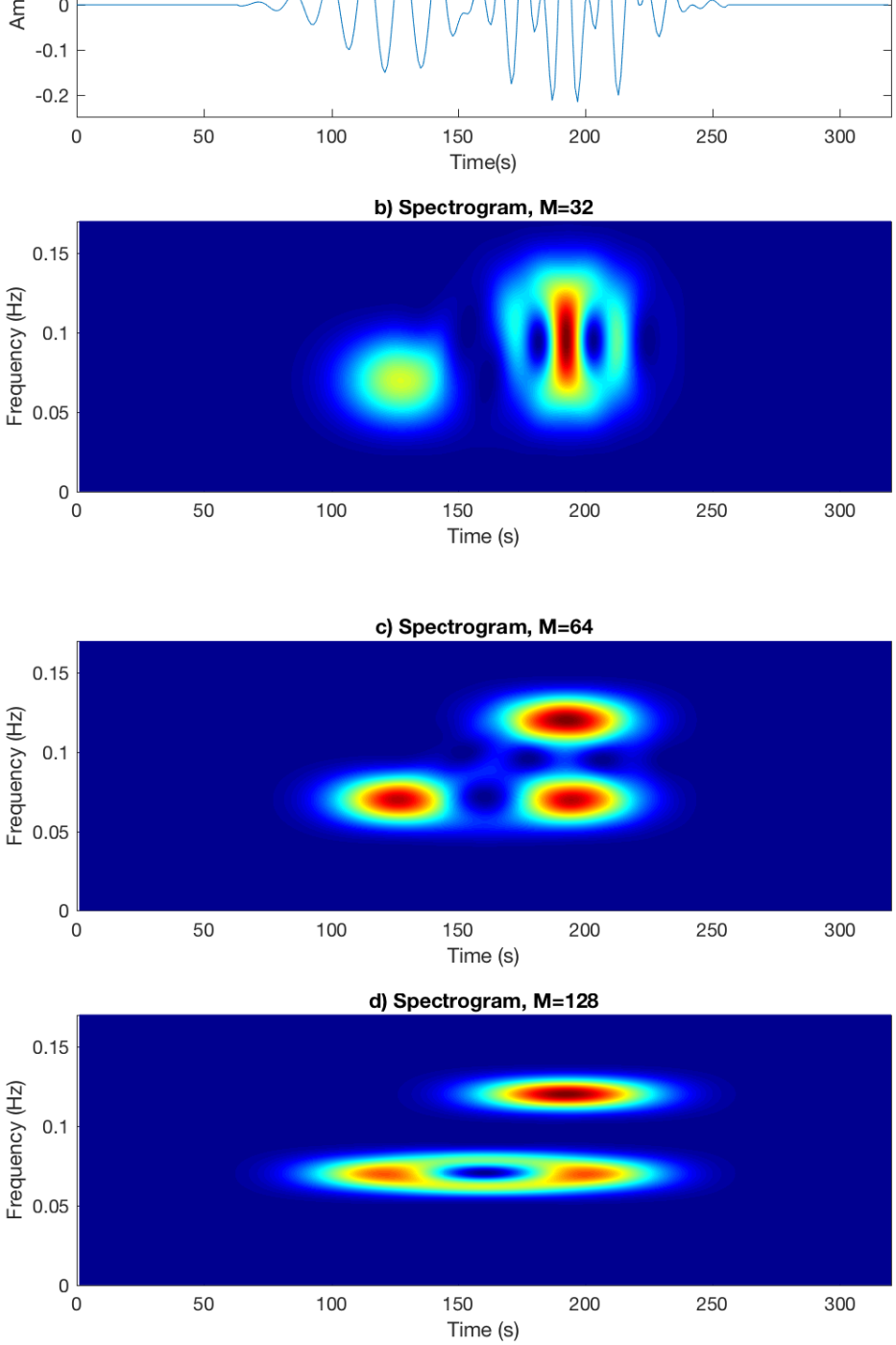
Window length of the spectrogram



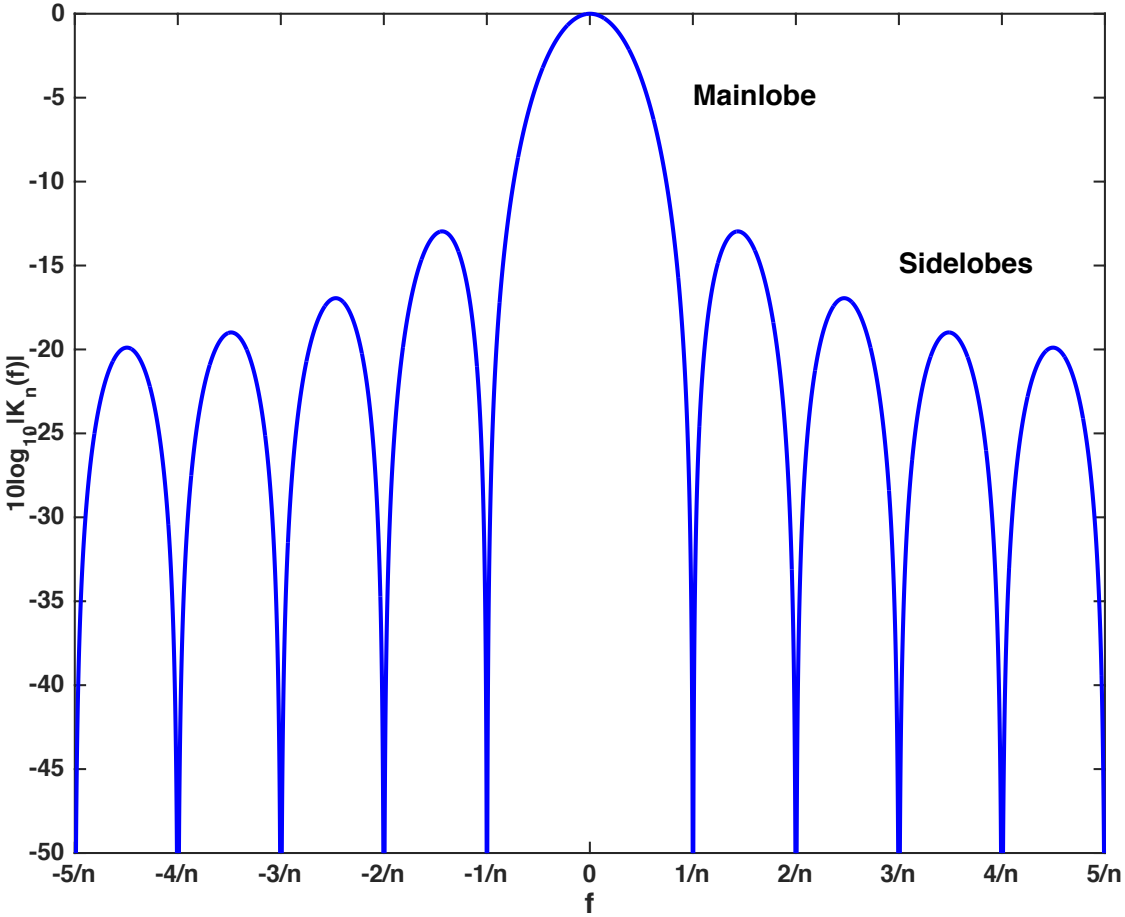
Window length of the spectrogram



Window leng

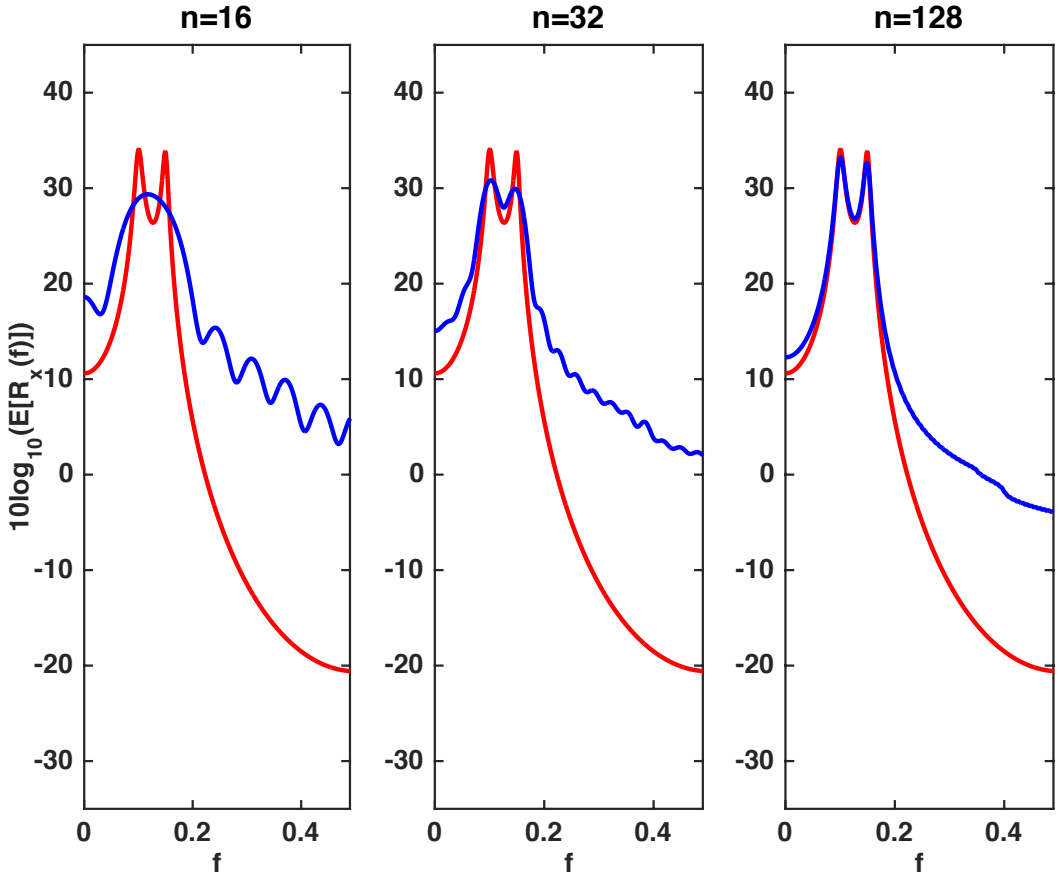

Am
0
-0.1
-0.2
0
50
100
150
200
250
300
Time(s)
b) Spectrogram, M=32
Frequency (Hz)
0.15
0.1
0.05
0
Time (s)
c) Spectrogram, M=64
Frequency (Hz)
Time (s)
d) Spectrogram, M=128
Frequency (Hz)
Time (s)

The window spectrum



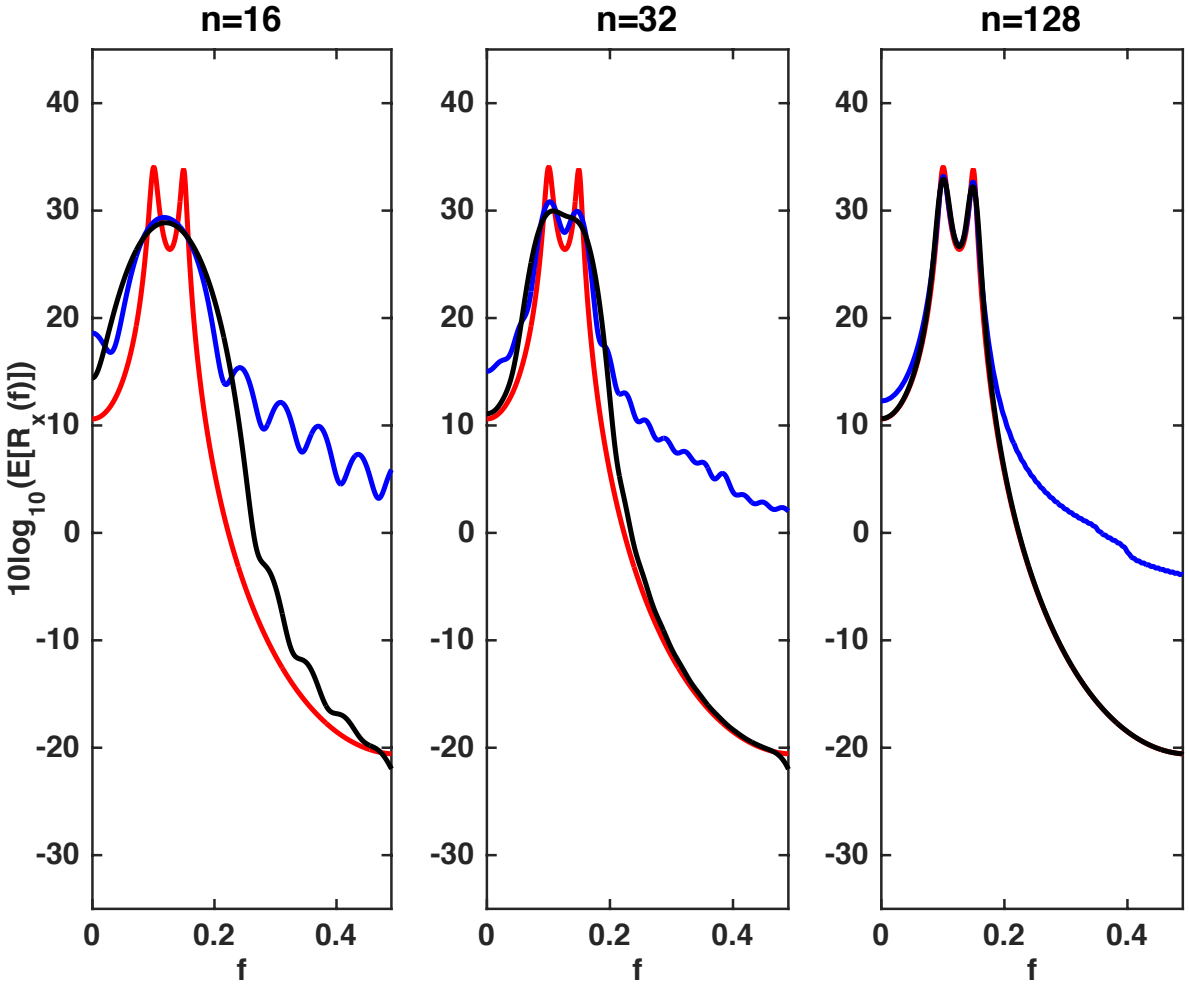
Example

Periodograms of an AR-process show the decreased resolution for small n and the large bias at the low-energy parts of the spectral density.

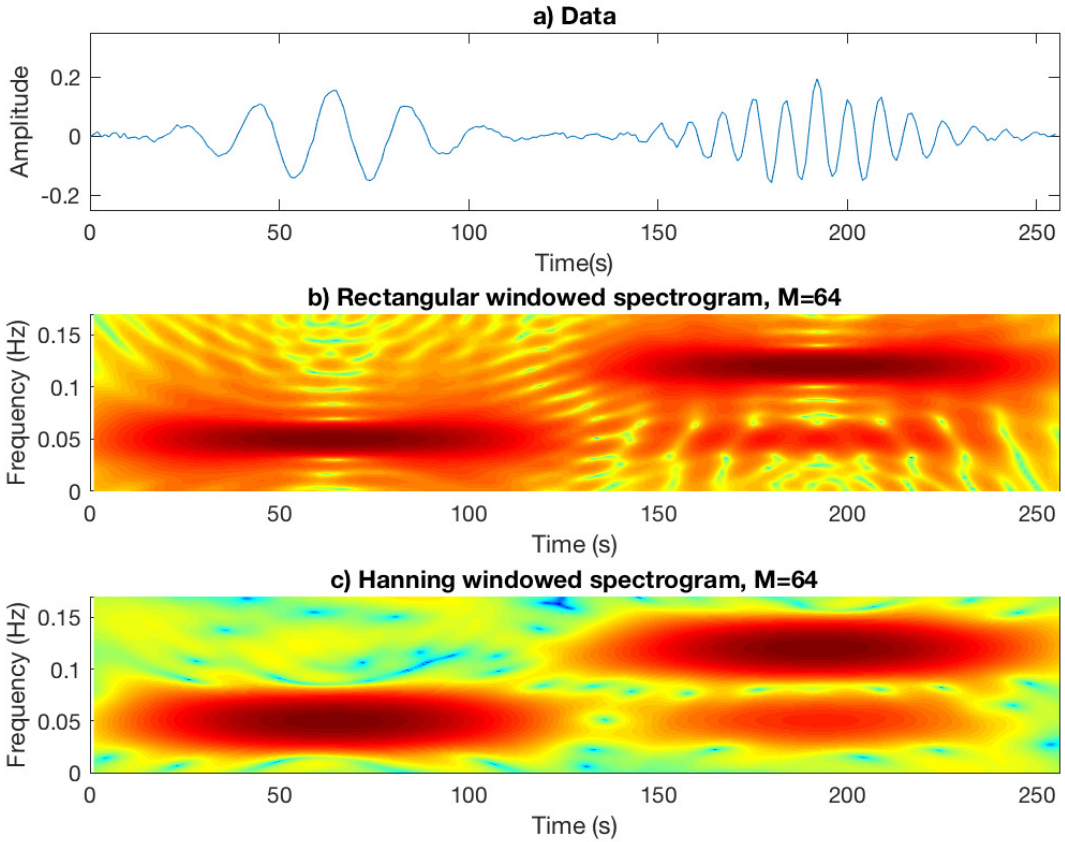


Example

Expected value of the spectral estimate using different lengths of the rectangle window (blue) and the Hanning window (black).



Rules for spectrogram window design



- The window length should match the component length.
- The window shape should be chosen for the best trade off between sidelobe suppression and mainlobe width.

The spectrogram concentration

With a Gaussian function as signal $x(t) = (\frac{\beta}{\pi})^{\frac{1}{4}} e^{-\frac{\beta}{2}t^2}$, and the Gaussian window, $h(t) = (\frac{\alpha}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha}{2}t^2}$, the spectrogram is

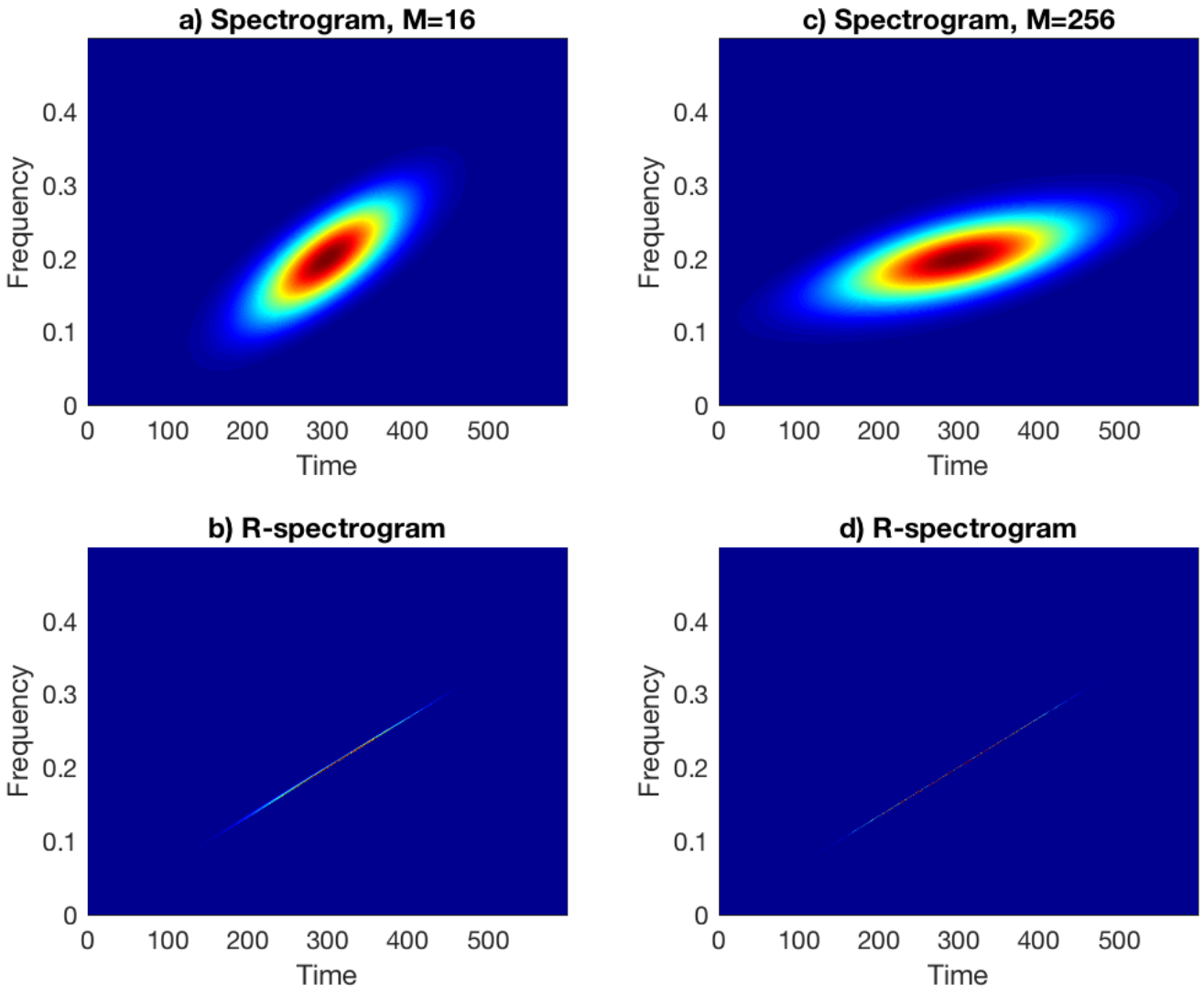
$$S_x(t,f) = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta} e^{-(\frac{\alpha\beta}{\alpha+\beta}t^2 + \frac{4\pi^2}{\alpha+\beta}f^2)}.$$

The area of the ellipse e^{-1} down from the peak value is

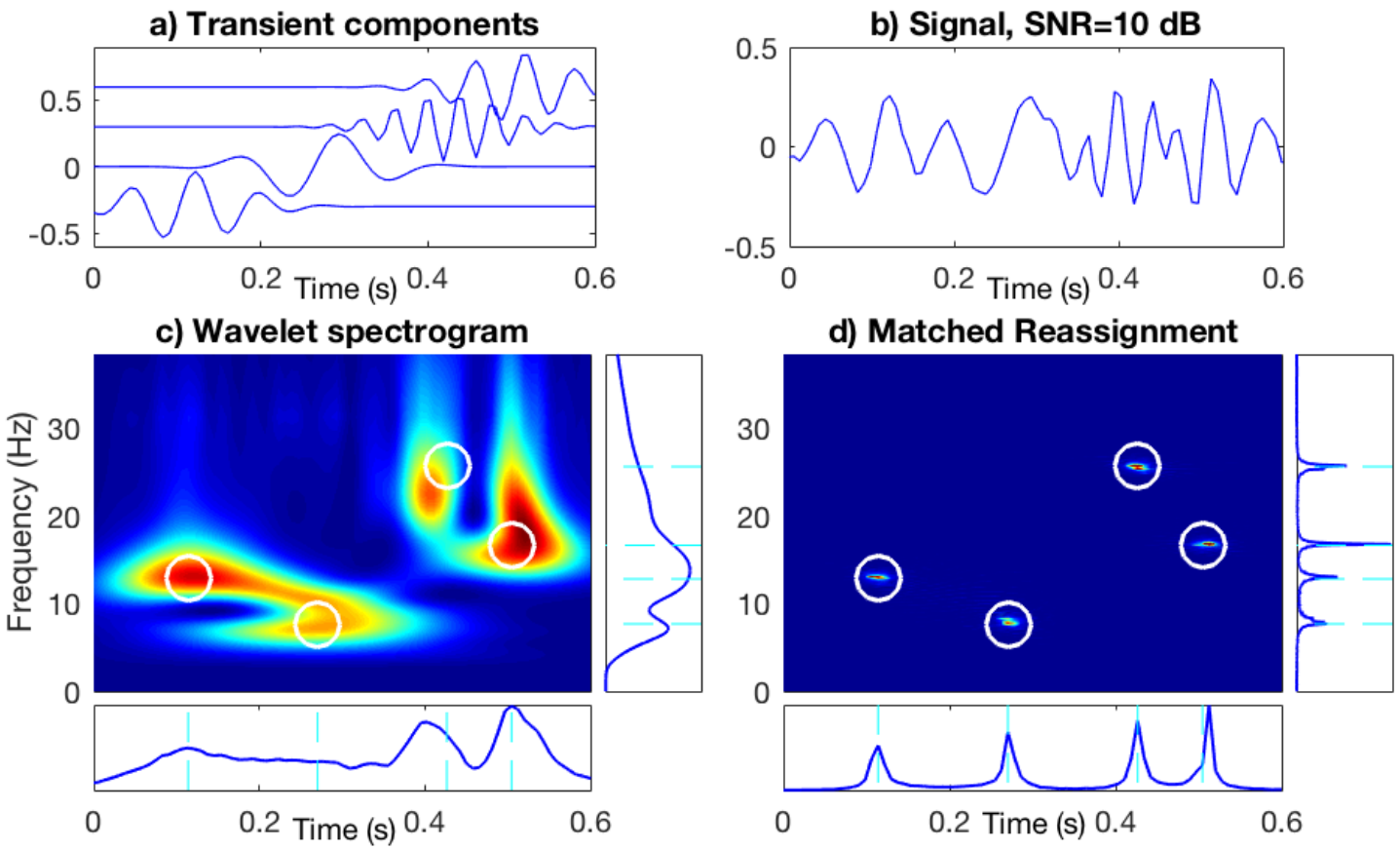
$$A = \frac{1}{2}\frac{\alpha+\beta}{\sqrt{\alpha\beta}}.$$

The minimum area $A = 1$ is found when $\alpha = \beta$.

Spectrogram reassignment



Matched reassignment



The Wavelet transform and the Scalogram

The definition of the continuous wavelet transform (CWT) is

$$CWT(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t_1) h^*(\frac{t_1 - b}{a})\, dt_1$$

where $h(t)$ is a continuous function in both the time domain and the frequency domain. The corresponding **scalogram** is defined as

$$SC_x(b,a) = |CWT(b,a)|^2.$$

Analysis and reconstruction

Gabor chose the Gaussian signal as **elementary function** in the Gabor expansion, as this function is the most concentrated both in time and frequency. The **Gabor transform** and the related **Wavelet transform** can be used for analysis and formulas for reconstruction can be derived if the time-frequency grid and window functions are chosen appropriately.

Analysis and reconstruction

Gabor expansion is defined as

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{m,k} g(t - mT_0) e^{i2\pi k F_0 t},$$

where $g(t)$ is the elementary function and T_0, F_0 denote the time and frequency sampling steps. The related sampled STFT, also known as the **Gabor transform**, is

$$X(mT_0, kF_0) = \int_{-\infty}^{\infty} x(t_1) w^*(t_1 - mT_0) e^{-i2\pi k F_0 t_1} dt_1,$$

where $X(mT_0, kF_0) = a_{m,k}$, if the sampling distances T_0 and F_0 satisfy the **critical sampling** relation $F_0 \cdot T_0 = 1$.

Analysis and reconstruction

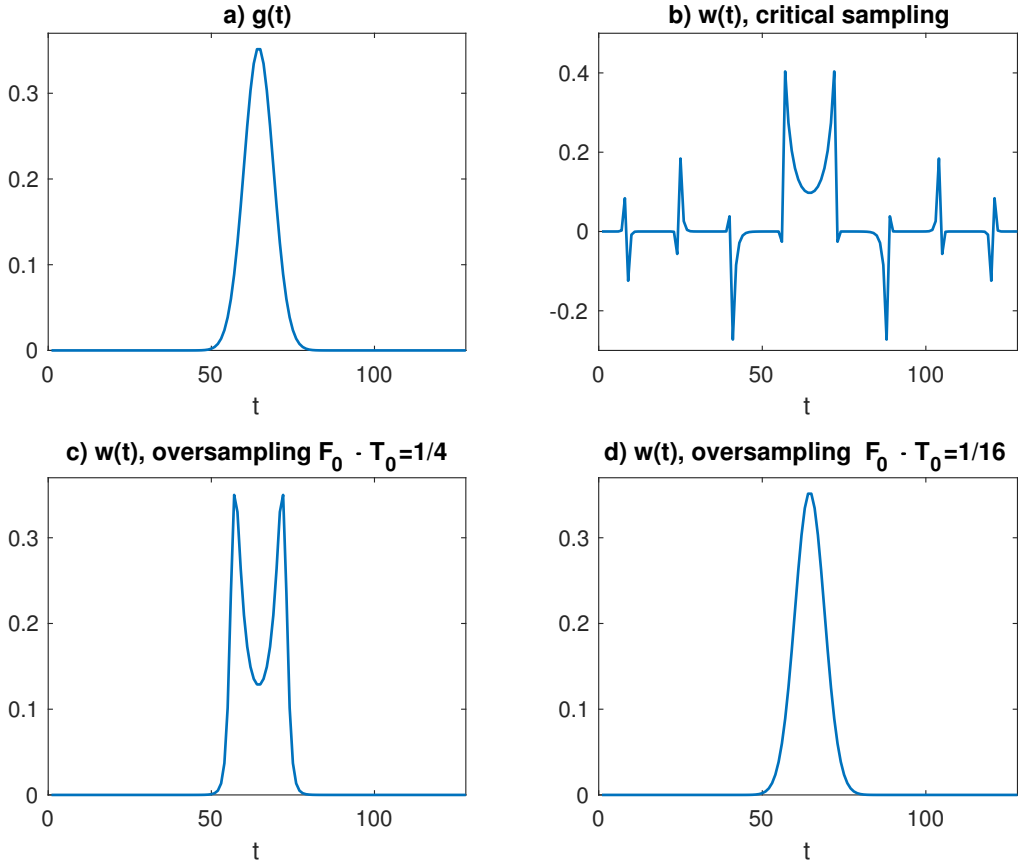


Figure: a) A Gaussian synthesis window $g(t)$. The analysis window $w(t)$ at; b) critical sampling; c) oversampling a factor 4; d) oversampling a factor 16.

Wigner distribution

The **Wigner distribution** is defined as

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-i2\pi f\tau}d\tau.$$

Eugene Wigner knew that there were other joint densities but he decided upon this: 'because it seems to be the simplest', (quote).

Usually, a real-valued signal $x(t)$ is replaced by the **analytic signal** correspondence $z(t)$ in the calculation of the **Wigner-Ville** distribution.

Wigner spectrum

For a non-stationary process, we define an **instantaneous autocorrelation function**, (IAF), as

$$r_x(t,\tau) = E[x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})].$$

As an extension of the Wiener-Khintchine theorem, the time-varying spectral density is given as

$$\begin{aligned} S_x(t,f) &= \int_{-\infty}^{\infty} r_x(t,\tau)e^{-i2\pi f\tau}d\tau, \\ &= \int_{-\infty}^{\infty} E[x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})]e^{-i2\pi f\tau}d\tau. \end{aligned}$$

This formulation is called the **Wigner spectrum**.

Stochastic time-frequency analysis

The fundamental problem in **stochastic** time-frequency analysis is the estimation of a reliable time-frequency spectrum from a single realization of the process. Different restrictions are therefore made of the non-stationarity. However, many of the general time-frequency representation methods can be used for successful estimation, although they are actually developed for deterministic signals.

The locally stationary process

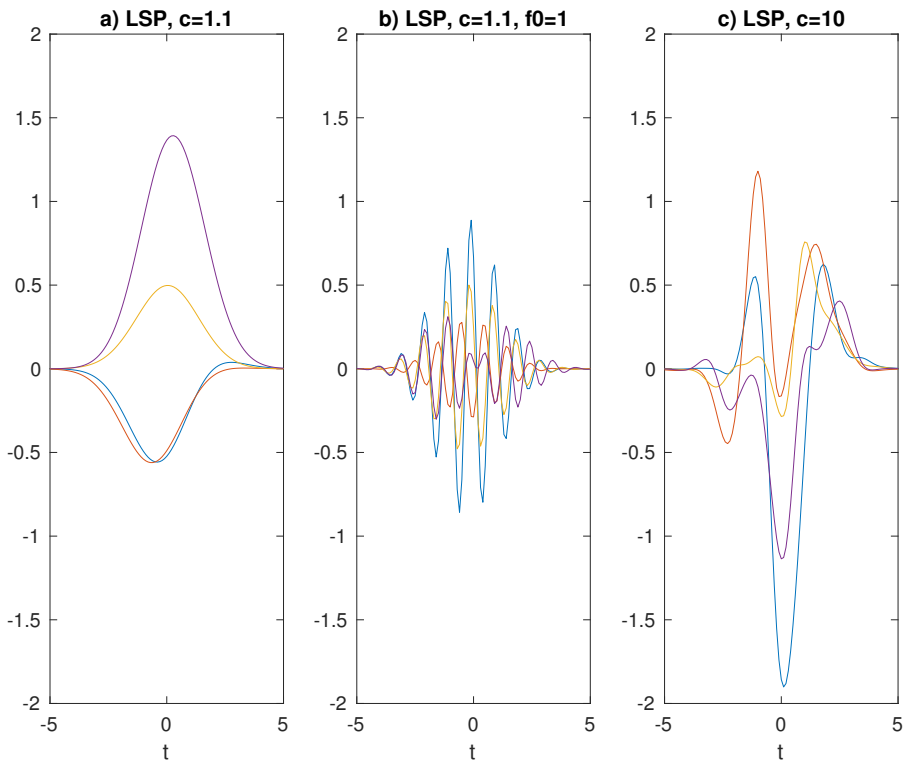


Figure: Examples of realizations of locally stationary processes where $q(t)$ and $r(\tau)$ are Gaussian functions.

Some special signals

The Wigner distribution of:

- $x(t) = e^{i2\pi f_0 t}$ is

$$W_x(t, f) = \delta(f - f_0).$$

- $x(t) = \delta(t - t_0)$ is

$$W_x(t, f) = \delta(t - t_0).$$

- $x(t) = e^{i\pi\beta t^2 + i2\pi f_0 t}$ is

$$W_x(t, f) = \delta(f - f_0 - \beta t).$$

where $-\infty < t < \infty$ and $-\infty < f < \infty$.

Properties of the Wigner distribution

- The Wigner distribution is **real valued** even if the signal is complex valued, i.e., $W_x(t, -f) = W_x(t, f)$. Assignment!
- The Wigner distribution is **time-shift and frequency-shift invariant**, i.e., if $y(t) = x(t - t_0)$ then

$$W_y(t, f) = W_x(t - t_0, f),$$

and if $y(t) = x(t)e^{i2\pi f_0 t}$ then

$$W_y(t, f) = W_x(t, f - f_0).$$

Assignment!

Properties of the Wigner distribution

- The Wigner distribution also satisfies the **marginals**, i.e.,

$$\int_{-\infty}^{\infty} W_x(t,f)df = |x(t)|^2,$$

and

$$\int_{-\infty}^{\infty} W_x(t,f)dt = |X(f)|^2,$$

where $X(f)$ is the Fourier transform of $x(t)$.

- If the marginals are satisfied the **total energy condition** is also automatically satisfied,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)dtdf = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = E_x.$$

Wigner distribution of a Gaussian signal

If the signal is

$$x(t) = (\frac{\beta}{\pi})^{\frac{1}{4}} e^{-\frac{\beta}{2}t^2},$$

the Wigner distribution is

$$W_x(t,f) = 2e^{-(\beta t^2 + \frac{4\pi^2 f^2}{\beta})}.$$

One measure of the concentration of the Wigner distribution is to compute the area of the cross-section ellipse at a specific height. We measure the area of the ellipse e^{-1} down from the peak value, ($W_x(0,0) = 2$) i.e. at the level $2e^{-1}$. The area of the cross-section $\beta t^2 + \frac{4\pi^2 f^2}{\beta} = 1$ is calculated as $A = \frac{1}{2}$.

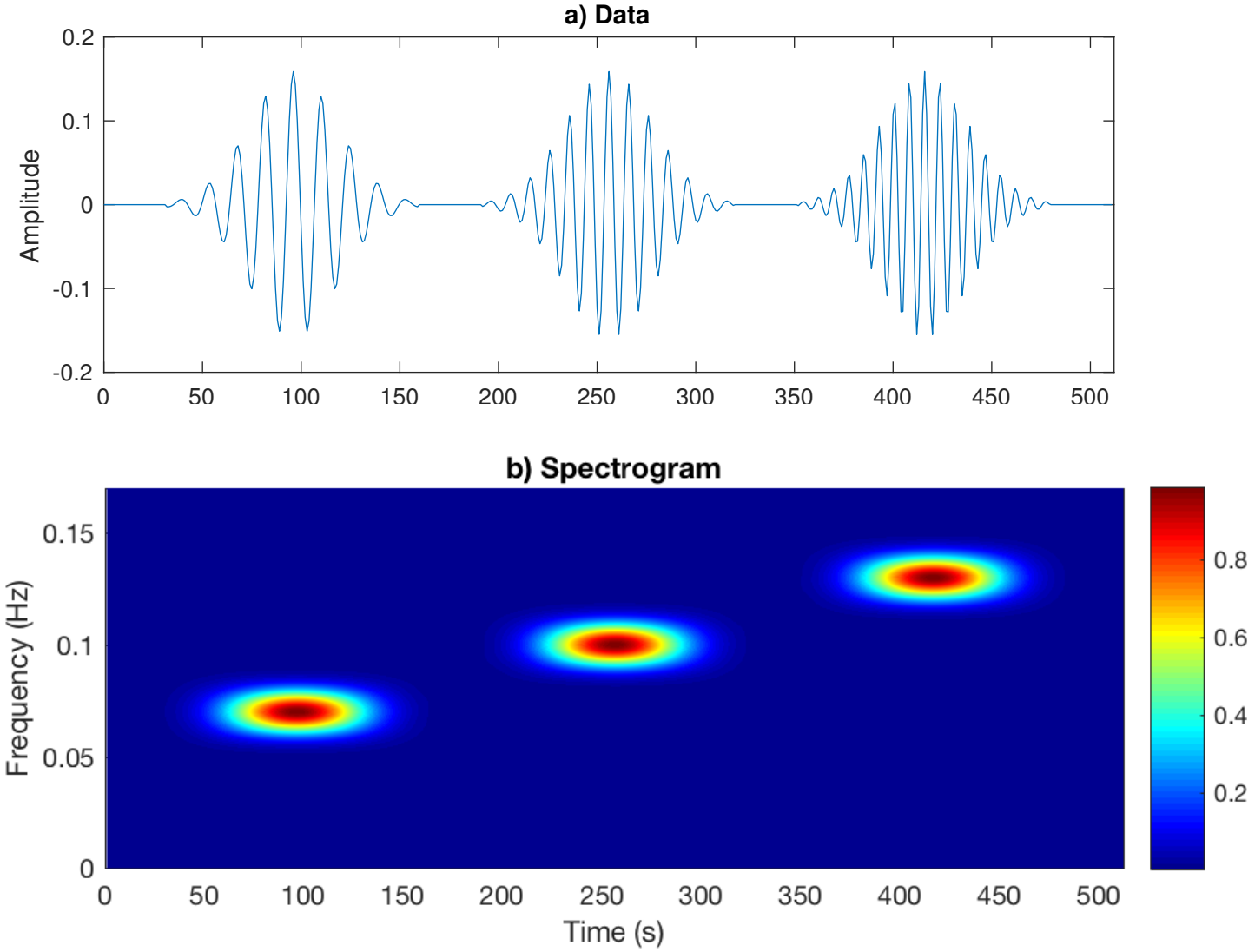
Cross-terms

We study the two-component signal $x(t) = x_1(t) + x_2(t)$ for which the Wigner distribution is,

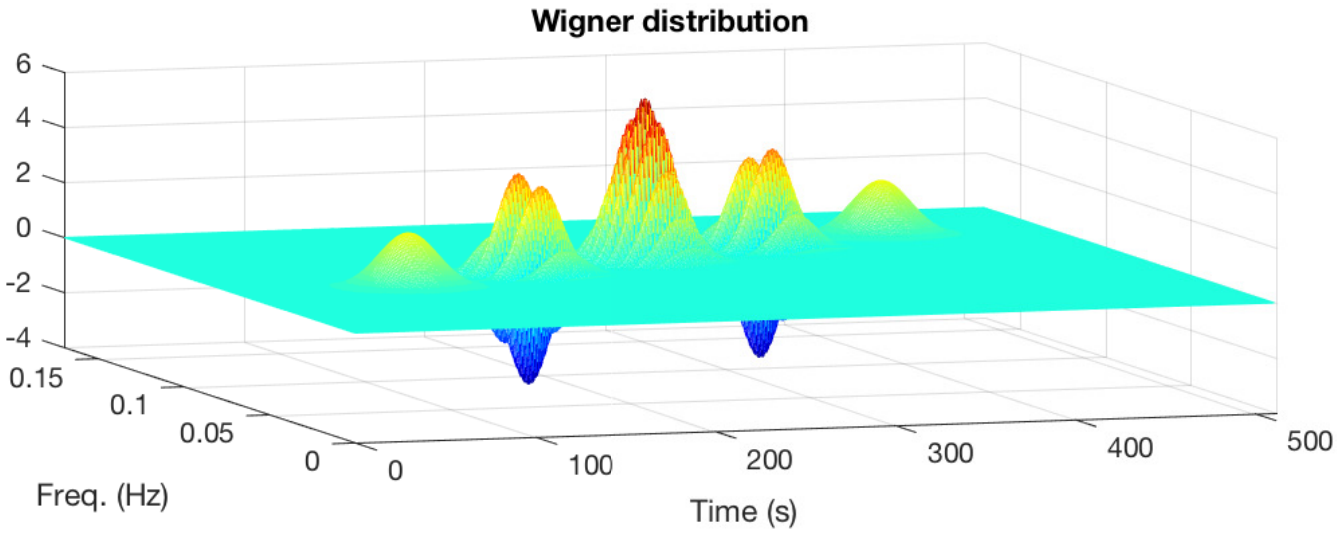
$$W_x(t, f) = W_{x_1}(t, f) + W_{x_2}(t, f) + 2\Re[W_{x_1,x_2}(t, f)],$$

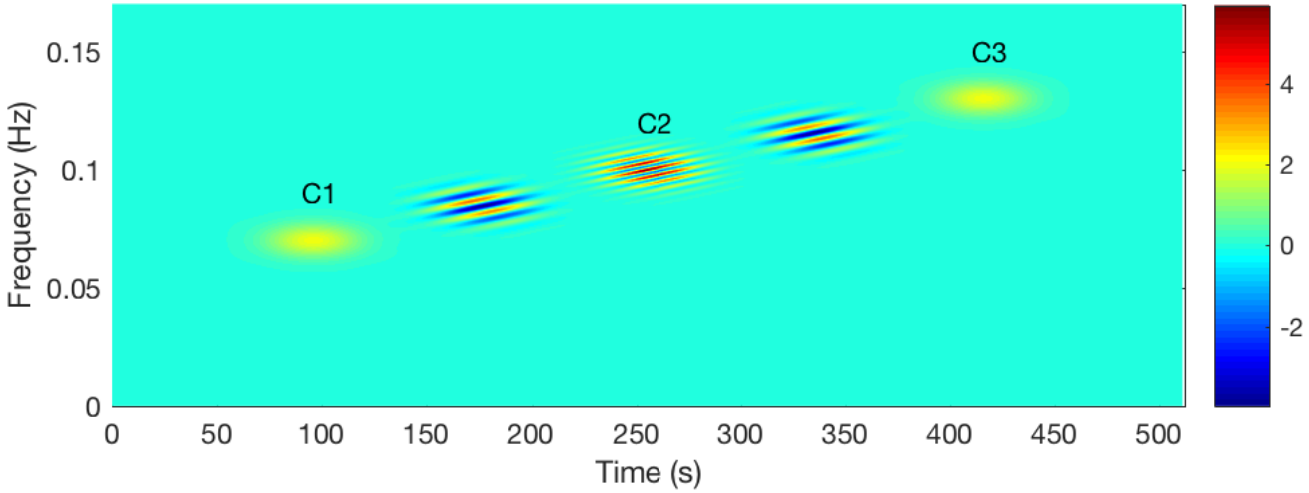
where $W_{x_1}(t, f)$ and $W_{x_2}(t, f)$), called **auto-terms**, are the Wigner distributions of $x_1(t)$ and $x_2(t)$ respectively. The **cross-term** is $2\Re[W_{x_1,x_2}(t, f)]$, where $\Re$ represents the real part.

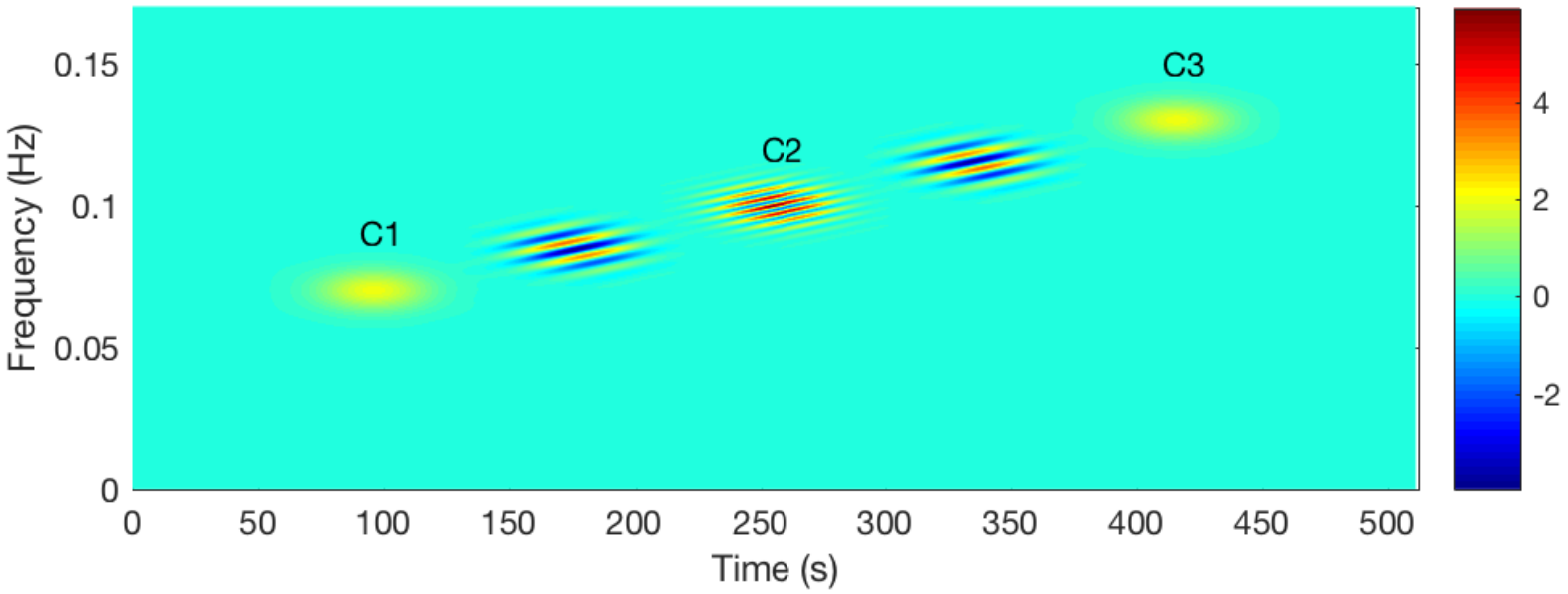
The spectrogram

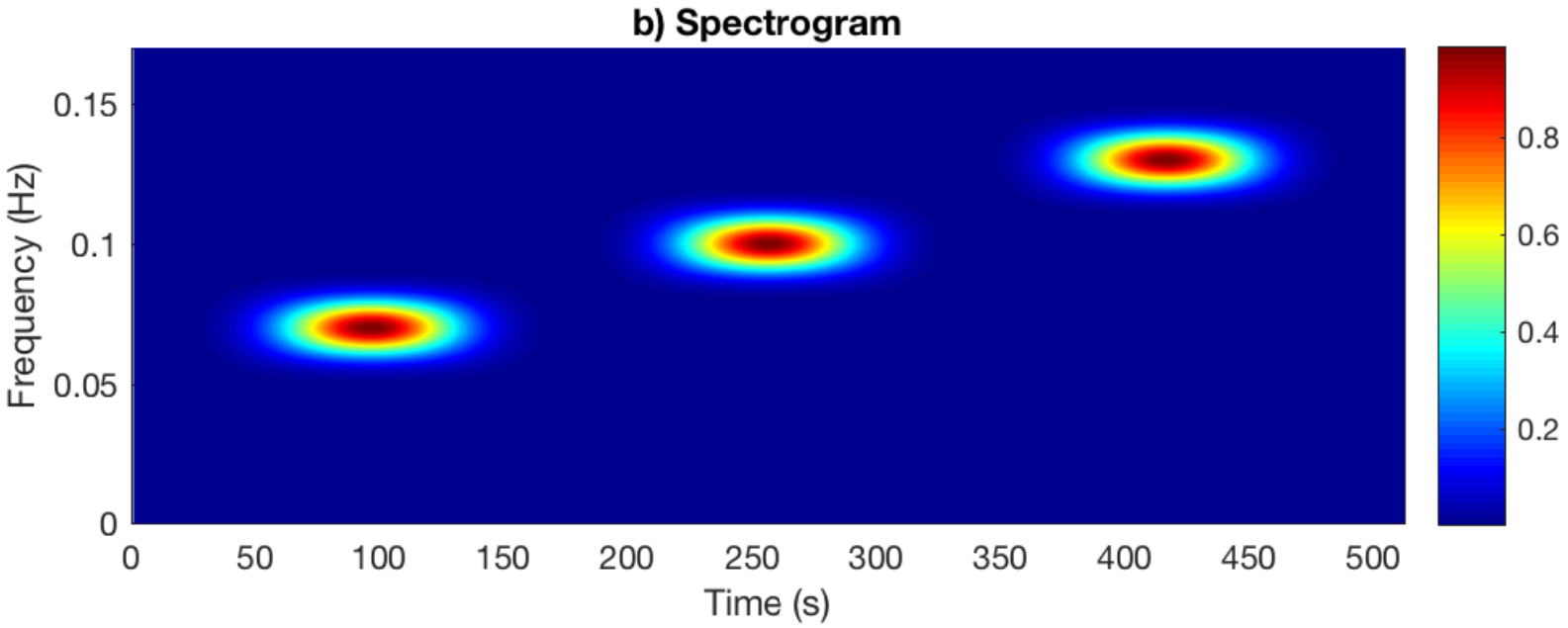


The Wigner distribution

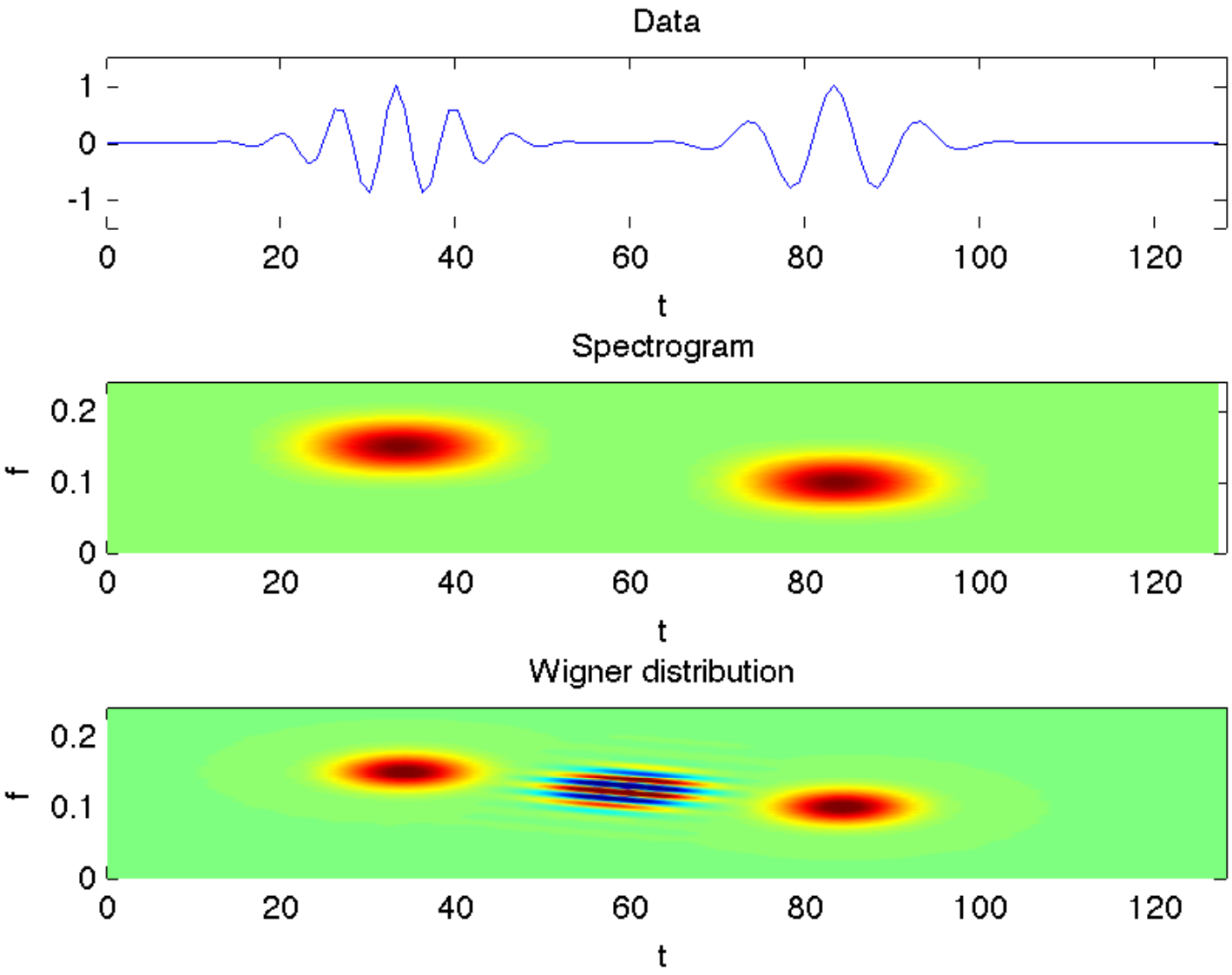





C1
C2
C3
Frequency (Hz)
0.15
0.1
0.05
0
0
50
100
150
200
250
300
350
400
450
500
Time (s)
4
2
0
-2


b) Spectrogram
Frequency (Hz)
0.15
0.1
0.05
0
0
50
100
150
200
250
300
350
400
450
500
Time (s)
0.8
0.6
0.4
0.2

Concentration and Cross-terms



Cross-terms

The cross-term:

- is twice as large as the two corresponding auto-terms.
- is always present and located midway between the auto-terms.
- is oscillating proportionally to the distance between the auto-terms.
- oscillation direction will be orthogonal to the line connecting the auto-terms.

Image enhancement

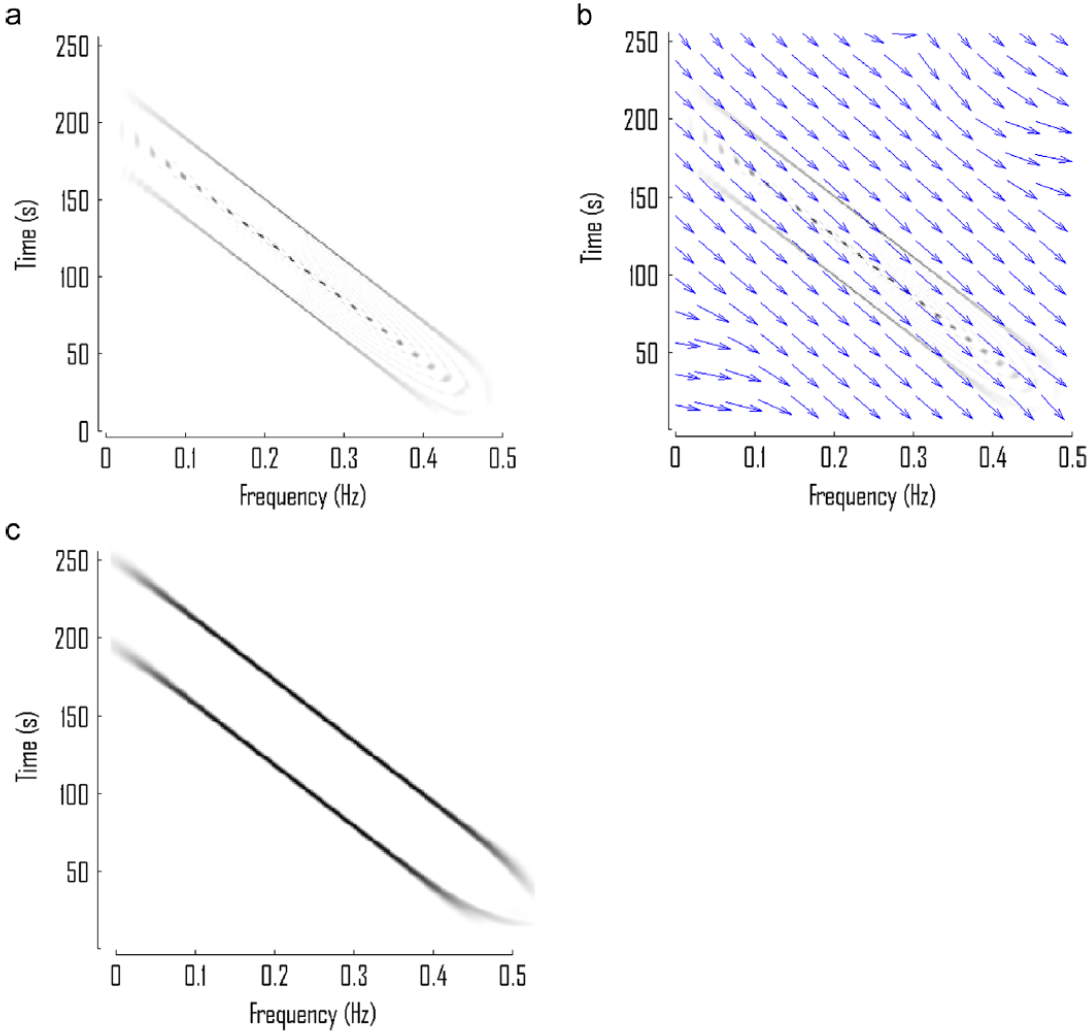
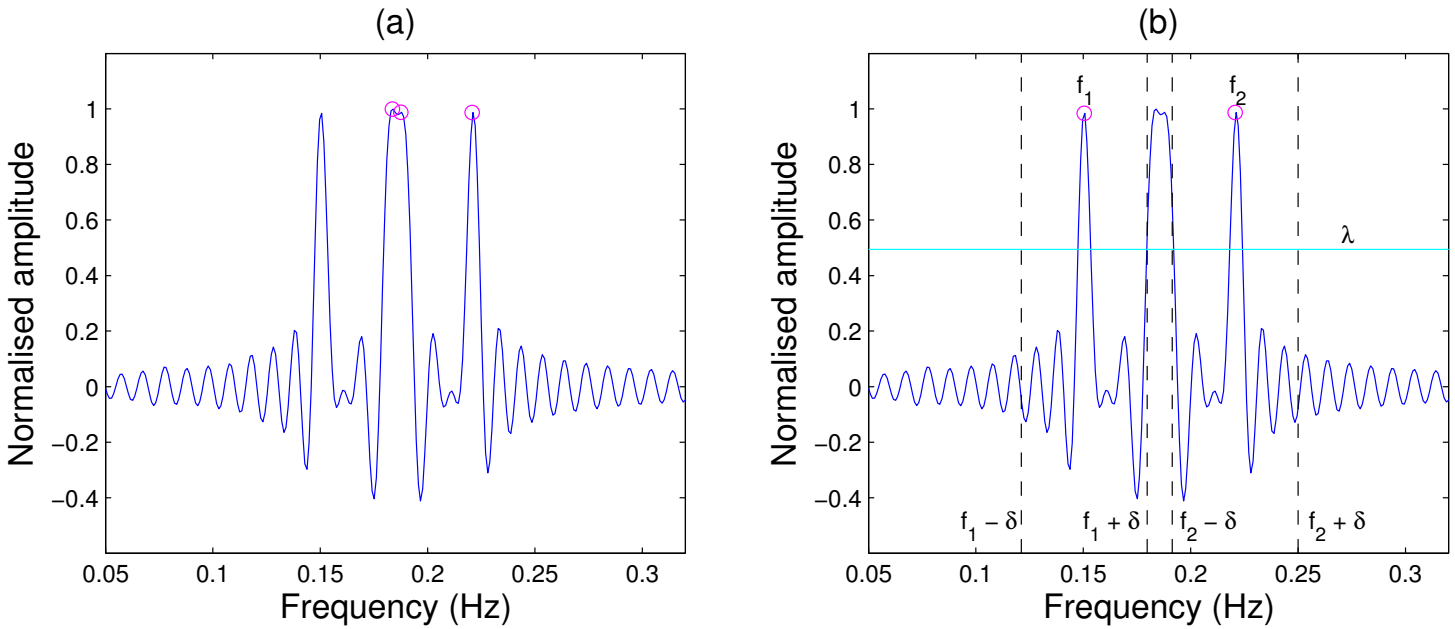


Fig. 1. Step by step Illustration. (a) The WVD. (b) The orientation image. (c). The DGFTFD.

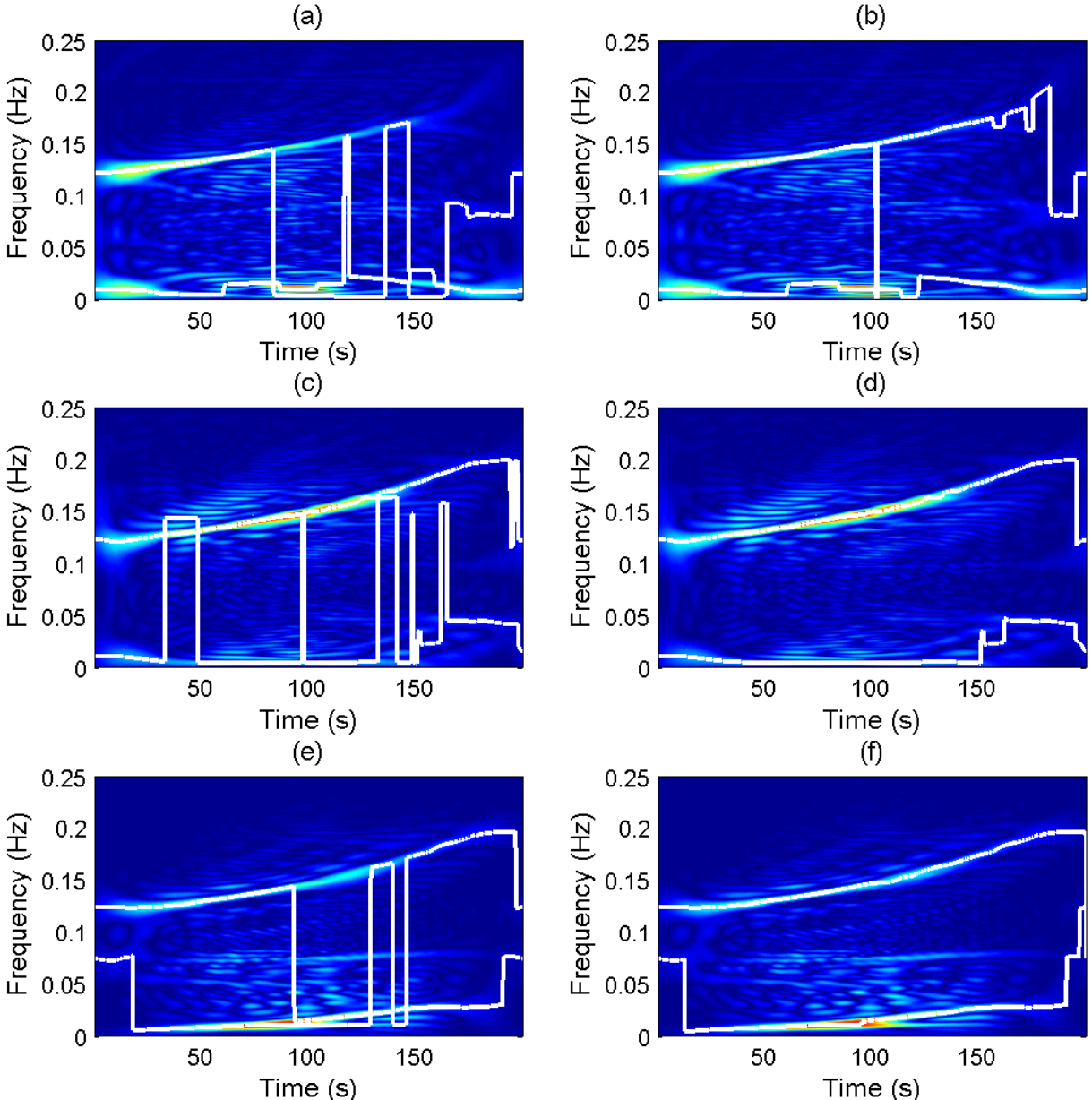
'Time-frequency image enhancement based on interference suppression in Wigner-Ville distribution', Nabeel Ali Khan, Maria Sandsten, Signal Processing, 2016.

Example



Isabella Reinhold and Maria Sandsten, 'Optimal time-frequency distributions using a novel signal adaptive method for automatic component detection' , Signal Processing, 2017.

Detection of heart rate variability



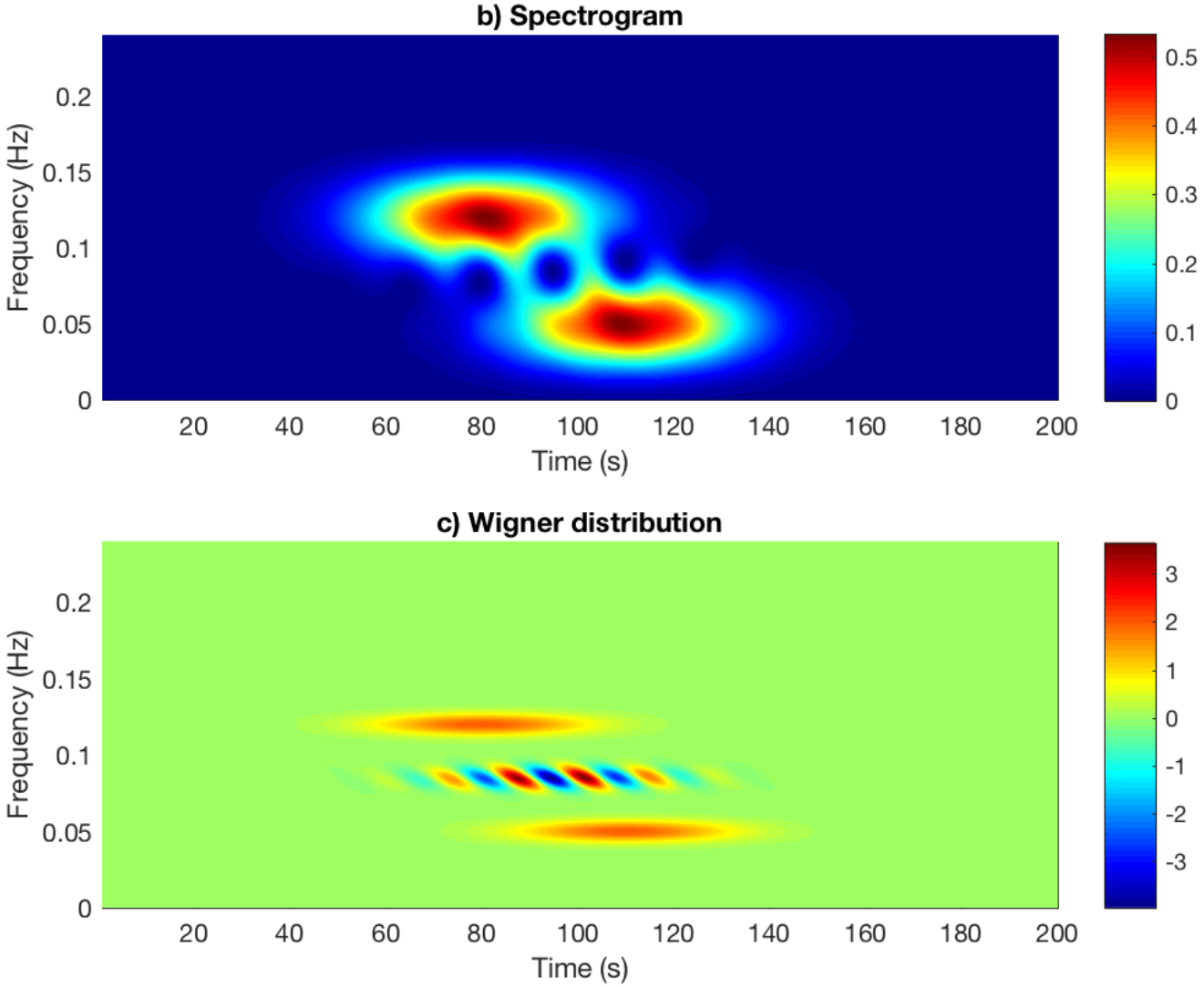
Cross-terms of the spectrogram

The spectrogram also give cross-terms but they show up as strong components mainly when the signal components are close to each other. This is seen in the following example:

$$\begin{aligned} |X(t,f)|^2 &= |\mathcal{F}\{x(t_1)h^*(t_1-t)\}|^2 = |X_1(t,f)+X_2(t,f)|^2 \\ &= |X_1(t,f)|^2 + |X_2(t,f)|^2 + 2\Re[X_1(t,f)X_2^*(t,f)], \end{aligned}$$

where the third term is included if the same time-frequency region is covered.

Cross-terms of the spectrogram



The discrete Wigner distribution

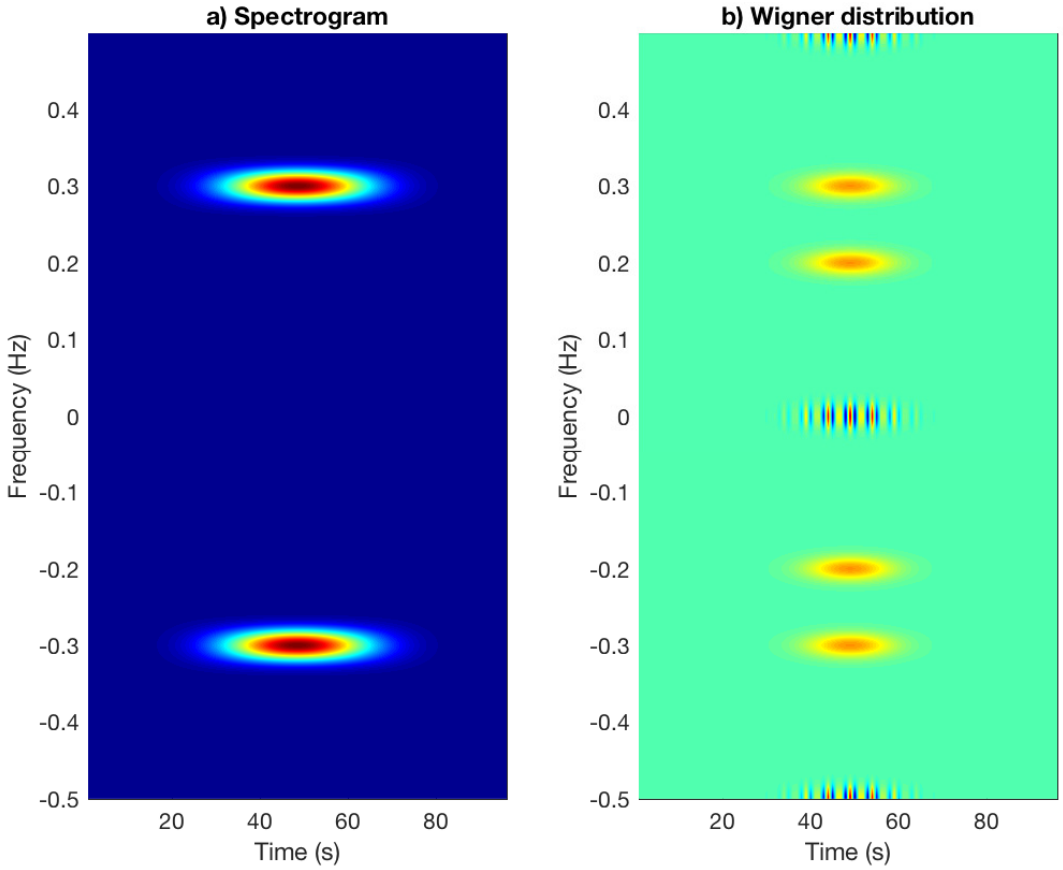
The discrete-time and discrete-frequency Wigner distribution is defined as

$$W_x[n,l] = 2 \sum_{m=-\min(n,N-1-n)}^{\min(n,N-1-n)} x_{n+m} x^*_{n-m} e^{-i2\pi m \frac{l}{L}}$$

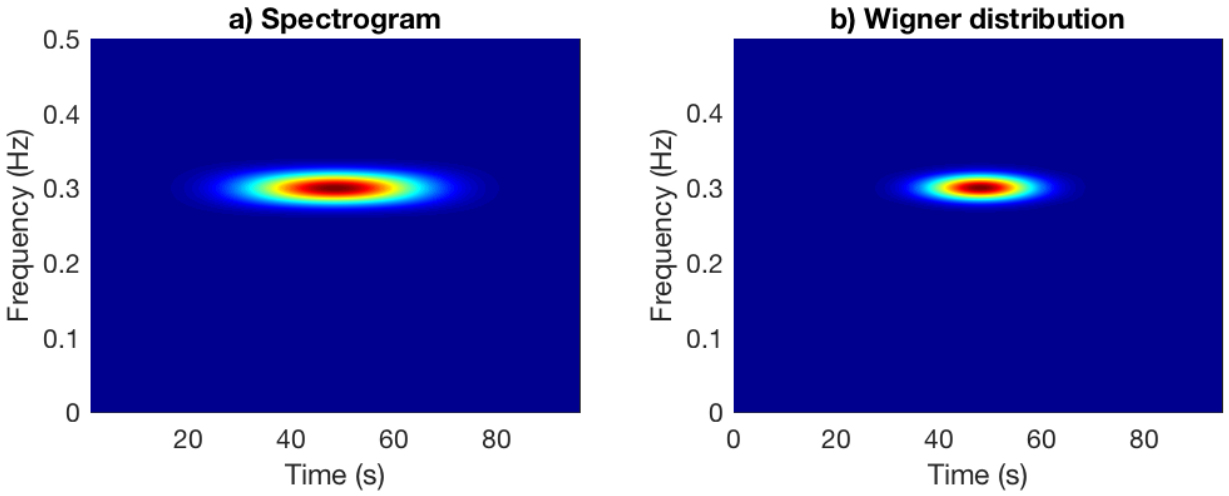
to be compared with

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\frac{\tau}{2}) x^*(t-\frac{\tau}{2}) e^{-i2\pi f \tau} d\tau.$$

The discrete Wigner distribution

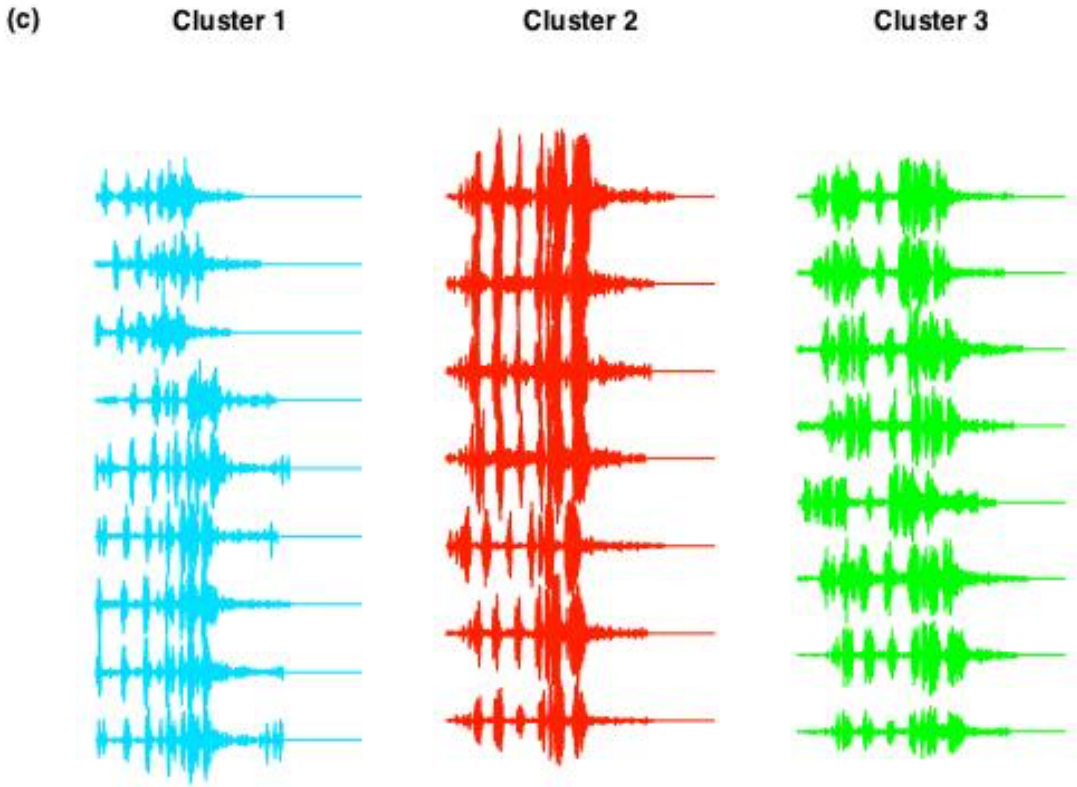


The discrete Wigner distribution

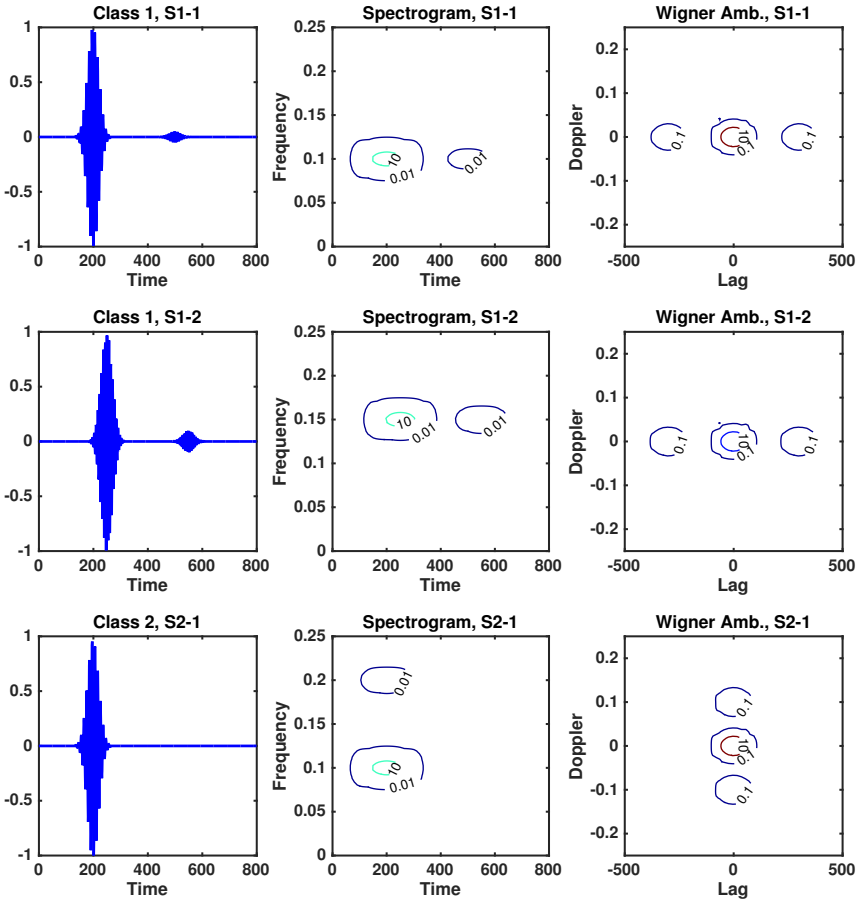


- To avoid **aliasing**, the highest normalized frequency of a **real-valued signal** could not be higher than $f_{max} = 0.25$.
- For the corresponding **analytic signal**, $f_{max} = 0.5$.

Classification of bird song syllables



Classification based on cross-terms



Maria Sandsten and Johan Brynolfsson, 'Classification of Bird Song Syllables Using Wigner-Ville Ambiguity Function Cross-Terms', EUSIPCO, Kos Island, Greece, 2017.

Classification based on cross-terms